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BUBBLE DISTRIBUTIONS IN A QUIESCENT OCEAN CALCULATED FROM THE BUBBLE TRANSPORT EQUATION

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Monterey, California

THESIS

BUBBLE DISTRIBUTIONS IN A QUIESCENT OCEAN CALCULATED FROM THE BUBBLE TRANSPORT EQUATION

by

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June 1972

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Bubble Distributions in a Quiescent Ocean Calculated from the Bubble Transport Equation

by

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ABSTRACT

A bubble transport equation was developed and a formal solution obtained for the bubble distribution function. The distribution function obtained depends upon the model used for bubble drag, gas diffusion, and ocean circulation. As a specific application, solutions were obtained for a one-dimensional, quiescent, steady state ocean with a bottom source at 197 meters. The characteristic equations were integrated numerically to test the accuracy of approximate analytical solutions. The calculated bubble densities were compared with experimental data available for this problem. When gas diffusion is neglected, the theoretical and experimental distributions are in general agreement.

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I. INTRODUCTION

Studies have been made by many researchers regarding the mechanisms of gas bubble dynamics in a fluid from both the theoretical [Ref. 1] and the experimental [Refs. 2, 3, and 4] aspects. The mechanisms of gas bubble formation, transport, and extinction are important as gas bubbles can effect the propagation of sound in the ocean by the underwater scattering of sound beams. Theoretical studies have explained how a bubble varies in size as it rises in a quiescent fluid. The phenomena involved includes shrinkage from gas diffusion versus expansion from decreasing hydrostatic pressure and shrinkage from surface tension for very small bubbles. However, there does not appear to be at present any theory explaining how the distribution of bubbles generated by a bottom or distributed source in the ocean can be expected to vary with depth or initial bubble radius.

This paper outlines the theory for determining theoretically the distributions of bubbles generated by a bottom source in the ocean [Ref. 5]. First, a linear bubble transport equation is developed and then it is solved by the method of characteristics. A model for a one-dimensional steady state ocean is specified for application of this theory. Depending on the assumptions made regarding the dynamics involved in the bubble motion, such as the importance of gas diffusion or surface tension, various semi-analytical solutions can be obtained for the distribution of bubbles from a bottom source. These distributions are then compared to available experimental data. In particular, the distributions from a bottom source located at 197 meters were calculated using various approximations and the results were compared with the experimental data in Ref. 4.

II. THE BUBBLE TRANSPORT EQUATION

The transport equation is basically a balance equation [Refs. 6 and 7]. It was desired to develop a transport equation for the description of the distribution of bubbles as a function of position, velocity, size (bubble radius) and time in a circulating field that may contain sources or sinks. A distribution function that describes the bubble population can be defined and if the bubbles satisfy the following general assumptions, this distribution will satisfy a Boltzmann-type transport equation:

(1) The bubbles are completely independent of one another;.

(2) If bubble interaction occurs, no more than two bubbles will interact at any one time.

Since bubbles in the ocean exact no force on one another and the history of one bubble does not affect the history of another bubble, the two above assumptions hold and a transport equation was found.

If $\psi(\vec{r},\vec{v},\ell,t)d^3rd^3vd\ell$ is the number of bubbles at time t with radius $d\ell$ about ℓ, velocity d^3v about $\vec{v}$, and position d^3r about $\vec{r}$, then at time $t' \equiv t + dt$, $\psi(\vec{r}',\vec{v}',\ell',t')\ d^3r'd^3v'd\ell'$ is the number of bubbles at time t + dt with radius $d\ell'$ about ℓ', velocity d^3v' about v', and position d^3r' about $\vec{r}'$, where $\vec{r}'$, $\vec{v}'$, and ℓ' are:

$$\begin{aligned}\vec{r}' &= \vec{r} + \vec{v}dt \\ \vec{v}' &= \vec{v} + \vec{a}dt \\ \ell' &= \ell + \nu dt\end{aligned} \tag{1}$$

The above Taylor series expansions about time t neglects all terms of order two and higher. The changes in position, velocity, and radius

are the result of nonzero velocity $\vec{v} \equiv d\vec{r}/dt$, acceleration $\vec{a} \equiv d\vec{v}/dt$, and radius change rate $\nu \equiv d\ell/dt$, respectively.

Since the transport equation is a balance equation the number of bubbles in a volume element $d^7\tau' \equiv d^3r'd^3v'd\ell'$ at time $t + dt$, must equal the number of bubbles that were in the volume $d^7\tau \equiv d^3rd^3vd\ell$ at time t plus or minus the bubbles that were generated or removed by any internal sources or sinks. Thus an equation for the conservation of bubbles is written as:

$$\psi(\vec{r}',\vec{v}',\ell',t+dt)d^7\tau' - \psi(\vec{r},\vec{v},\ell,t)d^7\tau = S(\vec{r},\vec{v},\ell,t)d^7\tau \qquad (2)$$

where $S(\vec{r},\vec{v},\ell,t)d^7\tau$ is the rate at which bubbles were introduced into the volume element $d^7\tau$ about $(\vec{r},\vec{v},\ell)$ by distributed sources.

Expanding $\psi' \equiv \psi(\vec{r}',\vec{v}',\ell',t+dt)$ in a Taylor series about $\psi \equiv \psi(\vec{r},\vec{v},\ell,t)$ and keeping only terms through the first order yields:

$$\psi' = \psi + (\partial\psi/\partial t)dt + (\vec{v} + \nabla_r\psi)dt + (\vec{a} \cdot \nabla_v\psi)dt + \nu(d\psi/\partial t)dt \qquad (3)$$

where

$$\vec{\nabla}_r(\cdot) \equiv \partial(\cdot)/\partial x,\ \partial(\cdot)/\partial y;\ \partial(\cdot)/\partial z$$

$$\vec{\nabla}_v(\cdot) \equiv \partial(\cdot)/\partial v_x,\ \partial(\cdot)/\partial v_y,\ \partial(\cdot)/\partial v_z\,.$$

The volume element $d^7\tau'$ was transformed into terms of $d^7\tau$ by using a Jacobian. The results of this transformation yielded

$$d^7\tau' = \frac{\partial(\vec{r}',\vec{v}',\ell')}{\partial(\vec{r},\vec{v},\ell)}\, d^7\tau \qquad (4)$$

with

$$\frac{\partial(\vec{r}',\vec{v}',\ell')}{\partial(\vec{r},\vec{v},\ell)} = 1 + (\vec{\nabla}_v \cdot \vec{a})dt + (\partial\nu/\partial\ell)dt\,. \qquad (5)$$

Substituting (3), (4), and (5) into (2), dividing by $d^7\tau dt$ and taking the limit dt → 0 yields the bubble transport equation [Ref. 5]:

$$\frac{\partial\psi}{dt} + \vec{v} \cdot \vec{\nabla}_r\psi + \vec{a} \cdot \nabla_v\psi + \nu\frac{\partial\psi}{dt} = - (\vec{\nabla}_v \cdot \vec{a} + \frac{\partial\nu}{\partial\ell}) \psi + S. \qquad (6)$$

It should be noted that this transport equation is different from the usual transport equation encountered because the term $(\vec{\nabla}_v \cdot \vec{a} + \partial\nu/dt)$ is generally nonzero as the bubble radius change rate, $\nu(\vec{r},\vec{v},\ell,t)$, and acceleration, $\vec{a}(\vec{r},\vec{v},\ell,t)$, are functions of bubble radius and bubble velocity respectively.

The distributed sources or sinks are contained in the term $S \equiv S(\vec{r},\vec{v},\ell,t)$. This term can also include the rate at which bubbles are introduced into $d^7\tau$ and removed from $d^7\tau$ by scattering off small scale turbulent eddies or by bubble-bubble interactions. It was assumed the S was independent of ψ or proportional to ψ. This simplication yielded a first order, partial differential bubble transport equation.

III. FORMAL SOLUTION TO THE TRANSPORT EQUATION

The bubble transport Eq. (6) in the form of a first order partial differential equation was solved using the method of characteristics [Ref. 5]. This was equivalent to replacing the transport equation with a set of simultaneous, first order, ordinary differential equations.

The characteristic equations were determined to be the bubble dynamic equations:

$$d\vec{r}/dt = \vec{v} \qquad \vec{r}(t_o) = \vec{r}_o = (x_o, y_o, z_o) \tag{7a}$$

$$d\vec{v}/dt = \vec{a} \qquad \vec{v}(t_o) = \vec{v}_o = (v_{xo}, v_{yo}, v_{zo}) \tag{7b}$$

$$d\ell/dt = \nu \qquad \ell(t_o) = \ell_o \tag{7c}$$

$$d\psi/dt = S - \Sigma_t\psi \qquad \psi(t_o) = \psi_o \,. \tag{7d}$$

The model chosen determined the acceleration $\vec{a}(\vec{r},\vec{v},\ell,t)$ and rate of bubble radius change $\nu(\vec{r},\vec{v},\ell,t)$, thus establishing the family of characteristic curves. This family was parameterized by the set of initial conditions $(\vec{r}_o,\vec{v}_o,\ell_o)$, and $\psi_o(\vec{r}_o,\vec{v}_o,\ell_o)$ was the initial distribution.

The general form of the distribution function was obtained by integrating Eq. (7d) and making use again of a Jacobian. The formal solution for the bubble distribution is [Ref. 5]:

$$\psi(\vec{r},\vec{v},\ell,t)d^7\tau = \psi(\vec{r}_o,\vec{v}_o,\ell_o,t)\ d^7\tau_o + \int_{t_o}^{t} S(\vec{r}'',\vec{v}'',\ell'',t'')d^7\tau''dt''$$

The integral here is a line integral along a characteristic curve as are all the integrals in this thesis. Equation (8) is equivalent to saying that the number of bubbles in $d^7\tau$ about $(\vec{r},\vec{v},\ell)$ at time t is the number of bubbles that were originally in $d^7\tau_o$ plus or minus those added or subtracted by sources or sinks while proceeding along the characteristic curve.

IV. THE MODEL FOR THE SOLUTION OF THE BUBBLE TRANSPORT EQUATION

Before solutions to Eq. (6) could be obtained explicitly, a model of the upper ocean had to be specified. This model must provide explicit expressions for the acceleration $\vec{a}(\vec{r},\vec{v},\ell,t)$, the rate of change of bubble radius $\nu(\vec{r},\vec{v},\ell,t)$, the bubble distribution $S(\vec{r},\vec{v},\ell,t)$ and the circulation field $\vec{V}(\vec{r},t)$. A Cartesian coordinate system was chosen with the origin located at the surface of the ocean and the z-axis vertical and positive upward. Thus all depths were taken to be negative.

A. THE BUBBLE RADIUS CHANGE RATE

The rate of change of bubble radius for bubbles circulating in the ocean is caused by either gas diffusion, a change in the hydrostatic pressure, surface tension, or a combination of these phenomena. This interaction of effects was analyzed by assuming the bubbles to be spherical containing an ideal gas. From the ideal gas law

$$V = \frac{4}{3}\pi\ell^3 = \frac{nRT}{P} \qquad (9)$$

where R is the gas constant, T is the temperature, P is the pressure, and V is the bubble volume. By taking the time derivative of Eq. (9) and assuming isothermal changes, an expression for ν, the change in bubble radius was obtained:

$$\nu \equiv \frac{d\ell}{dt} = \frac{\ell}{3}\left[\frac{1}{n}\frac{dn}{dt} - \frac{1}{P}\frac{dP}{dt}\right]. \qquad (10)$$

The term $\frac{1}{n}\frac{dn}{dt}$ contains the gas diffusion effects while the effects of the change in hydrostatic pressure and surface tension are contained in the term $\frac{1}{P}\frac{dP}{dt}$.

The hydrostatic pressure P is the sum of the atmospheric pressure, P_o, and the fluid pressure, $-\rho_o gz$. The surface tension pressure is $2\gamma/\ell$, where ρ_o is the density of sea water, $\rho_o \cong 1.03 \times 10^3$ Kg/meter^3, and γ is the surface tension of sea water, approximately equal to 0.74 newtons/meter. Therefore

$$P = P_o - \rho_o gz + 2\gamma/\ell \tag{11}$$

and

$$\frac{dP}{dt} = - \rho_o g v_z - \frac{2\gamma}{\ell^2}\frac{d\ell}{dt} . \tag{12}$$

By using a simple model for gas diffusion [Ref. 8] the following expression was obtained for dn/dt

$$\frac{dn}{dt} = - \delta 4\pi\ell^3 (P - P_o) \tag{13}$$

where δ is a constant. A value for δ had been determined from experimental data to be about 10^{-6} meters/second. It was assumed for this model that the dissolved gas concentration gradient was confined to a thin shell at the air-water boundary because of the bubble's speed.

An expression for ν was obtained by substituting Eqs. (11), (12), and (13) into Eq. (10) [Ref. 5].

$$\nu \equiv \frac{d\ell}{dt} = [v_z\ell^2 + 3\delta RT(z\ell - \Gamma)]/[3(D-z)\ell + 2\Gamma] \tag{14}$$

where

$$\Gamma \equiv 2\gamma/\rho_o g \simeq 1.47 \times 10^{-5} \text{ meters}^2$$

and

$$D \equiv P_o/\rho_o g \simeq 10 \text{ meters.}$$

If ℓ is greater than 30 microns (1 micron (μ) $= 10^{-6}$ meters), the surface tension terms can be ignored, i.e., $\Gamma = 0$.

B. THE BUBBLE EQUATION OF MOTION AND ACCELERATION

A bubble that is in motion with respect to a stationary reference system will experience several forces acting on it. First there is the force of the water acting on the bubble to pull it along. If the bubble were completely entrained by the fluid so that the bubble's velocity ($\vec{v}$) were equal to the circulation field velocity ($\vec{V}$), the entrainment force would be $\rho_o \sigma d\vec{V}/dt$ where σ is the volume of the bubble. Complete entrainment can occur only if ρ is equal to ρ_o. Since this is rarely the case, the bubble will maintain a relative velocity $\vec{u} = \vec{v} - \vec{V}$ with respect to the water. This relative velocity produces a drag force, $\vec{F}_D$, acting on the bubble to slow it down.

As the bubble moves through the water, the bubble pulls water along with it. The amount of water pulled along with it is equal to some fraction, β, of the bubble's volume. From experimental data β is usually taken as 0.5. This phenomena is called the mass adhering force and is equal to $-(\beta\sigma\rho_o d\vec{u}/dt)$. The bubble will also experience a net bouyant force which is equal to $\sigma g(\rho_o - \rho)\,\hat{z}$, where g is the gravitational acceleration.

By using Newton's second law these forces were combined to yield an expression for the acceleration $\vec{a}$ [Ref. 5]:

$$\vec{a} = \frac{\vec{F}_D}{\sigma\rho_o(\beta+\rho/\rho_o)} + \frac{1+\beta}{(\beta+\rho/\rho_o)}\frac{d\vec{V}}{dt} + \frac{(1-\rho/\rho_o)}{(\beta+\rho/\rho_o)} g\hat{z} \qquad (15)$$

where

$$\frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + (\vec{v}\cdot\vec{\nabla}_r)\vec{V} \, . \qquad (16)$$

In order to obtain an explicit expression for the acceleration $\vec{a}$, it was necessary to derive an expression for the drag force $\vec{F}_D$. Various

models for the drag force were considered, with the models being classified by using the Reynold's number $R_e = 2\ell u\rho_o/\eta$ where η is the viscosity of water ($\eta \div 10^{-3}$ Newton-second/meter^2).

For bubbles with a radius less than or equal to 100 microns ($\ell \leq 100$), the Reynold's number is less than one ($R_e < 1$) and the drag is viscous according to Stokes Law:

$$\vec{F}_D \simeq \vec{F}_{Stokes} = -K\eta\ell\vec{u} \qquad (17)$$

where $K = 6\pi$ in the presence of surface active material and 4π in the absence of these surface active materials.

The drag was considered to be both viscous and turbulent if $100\mu \leq \ell \leq 1500\,\mu$ or $1 < R_e < 700$. Thus the drag force was

$$\vec{F}_D = \vec{F}_{Stokes} + \vec{F}_{Turb}\,. \qquad (18)$$

Here $\vec{F}_{Stokes}$ was given as in Eq. (17) with $6\pi \leq K \leq 12\pi$ and $\vec{F}_{Turb}$, which arise from the energy dissipated in the turbulence of the wake, had the form:

$$\vec{F}_{Turb} \simeq -0.5\,K_f\rho_o s_1\,u\vec{u} \qquad (19)$$

where K_f is a drag coefficient and s_1 is the bubble surface area covered by the turbulent wake ($s_1 = \pi\ell^2$). If there were no surface active materials present $\vec{F}_D$ would equal $\vec{F}_{Stokes}$, with $K = 12\pi$.

Since the oceans contain surface active materials, the drag force contains both a viscous drag and turbulent drag term. By noting that $\rho/\rho_o \ll 1$ and that $\sigma = 4\pi\ell^3/3$ for bubbles with $\ell < 1500\,\mu$, the acceleration can be written as [Ref. 5]:

$$\vec{a} = -\left(\frac{\alpha}{\ell^2}\right)\vec{u} - \left(\frac{\xi}{\ell}\right)u\vec{u} + \left(\frac{g}{\beta}\right)\hat{z} + \left(1 + \frac{1}{\beta}\right)d\vec{V}/dt \qquad (20$$

with

$$\alpha = \frac{9}{2}\frac{\eta}{\beta\rho_o} = 8.75 \times 10^{-6} \text{ meters/sec}^2$$

and

$$\xi = \frac{3}{2}\frac{K_f}{\beta}\frac{s_I}{4\pi\ell^2} = 4.5 \times 10^{-1} .$$

If the bubbles were generated by a surface source such as rain, wave action, etc., the bubble radii are generally less than 150 μ [Refs. 3 and 9] and the drag force is predominately viscous. In this case ξ should be set equal to zero.

If the bubbles originate at a bottom source, the radii are in the range of 500 μ to 700 μ [Ref. 4] and the drag should be considered to be predominately turbulent with α set equal to zero.

C. BUBBLE SOURCES

The bubbles in the ocean might physically be considered to have originated from three types of sources. The first type included bubbles that originated near the surface and were caused by wave action, rain or snow. The distribution of bubbles originating from these phenomena was analyzed in Ref. 10.

The second type of bubbles are those nucleated by distributed particles of material as well as bubbles that originated from biological activity such as photosythesis or bacteria and are distributed throughout the medium.

The third type of bubbles includes those bubbles generated on the floor of the ocean. These bubbles are often sediment-initiated bubbles and measurements indicate these bubbles to be between 450 μ -- 800 μ in radius with a peak around 550 μ [Ref. 4]. It was this category of bubbles for which the distribution function was evaluated. It was

considered to be mathematically convenient to introduce the bubbles via boundary conditions rather than to include them in $S(\vec{r},\vec{v},\ell,t)$. $\psi_o(\vec{r}_o,\vec{v}_o,\ell_o)$ represented the bubble distribution originating in the bottom sediment.

The model for the circulation field was chosen to be a quiescent fluid dictating that $\vec{V} = 0$. For this model the bubbles could rise only via buoyancy and the characteristic curves are vertical upward.

V. RESULTS

For this thesis the bubble transport theory was applied to a one-dimensional quiescent, steady state ocean with a bottom source at 197 meters. The solution and approximate solutions are given. The characteristic equations were integrated numerically to test the accuracy of the approximate analytical solutions. The calculated bubble distributions were compared with experimental data.

In this steady state model the velocities are vertical upward, and all functions depend spatially on depth only. The bubble transport equation (Eq. 6) can be written as:

$$v \frac{d\phi}{dz} + a \frac{d\phi}{dv_z} + \nu \frac{d\phi}{d\ell} = - \Sigma_t \phi + S. \tag{21}$$

Here $\phi(z,v,\ell)$ is the distribution function, S is the distributed source function, v is the vertical velocity, ν is the rate of change of bubble radius, and $\Sigma_t \equiv \partial a/\partial v + \partial \nu/\partial \ell$.

If z is chosen as the independent parameter, the characteristic equations are [Ref. 5]:

$$\frac{dv}{dz} = \frac{a}{v} = - \frac{\alpha}{\ell^2} - \frac{\xi}{\ell} v + \frac{g}{\beta v} \tag{22a}$$

$$\frac{d\ell}{dz} = \frac{\nu}{v} = \frac{\ell^2 + 3\delta RT(z\ell - \Gamma)/v}{3(D-z)\ell + 2\Gamma} \tag{22b}$$

and

$$\frac{d\phi}{dz} = - \frac{1}{v} \Sigma_t \phi + \frac{1}{v} S. \tag{22c}$$

Since the characteristic curves are vertically upward along the z-axis, the characteristics or boundary conditions are $v = v_o$, $\ell = \ell_o$, and $\phi = \phi(z_o, v_o, \ell_o)$ at $z = z_o \leq 0$.

Equation (22c) was integrated to yield

$$\phi(z,v,\ell) = \phi(z_o,v_o,\ell_o)\exp\left[-\int_{z_o}^{z}\Sigma_t(z',v',\ell')dz'/v'\right] + \int_{z_o}^{z}\exp\left[-\int_{z'}^{z}\Sigma_t(z'',v'',\ell'')dz''/v''\right]S(z',v',\ell')dz'/v' \qquad (23)$$

with the integration taking place along the characteristic curves $\{z,v(z;z_o,v_o,\ell_o),\ell(z;z_o,v_o,\ell_o);\ z \leq 0\}$. By using the Jacobian Eq. (23) was rewritten as [Ref. 5]:

$$v\phi(z,v,\ell)dvd\ell = v_o\phi(z_o,v_o,\ell_o)dv_od\ell_o + \int_{z_o}^{z}S(z',v',\ell')dz'dv'd\ell'. \qquad (24)$$

Equation (24) is interpreted as a conservation equation which states that the flux of bubbles, $v\phi$, at the point (z,v,ℓ) is equal to the flux of bubbles, $v_o\phi_o$, at the point (z_o,v_o,ℓ_o) plus (or minus) any bubbles added (or substracted) by distributed sources along the characteristic curve.

Prior to the calculation of the distribution function in either Eq. (23) or Eq. (24) it was necessary to establish a mapping between points (z,v,ℓ) and (z',v',ℓ') on the characteristic curve. This was done by integrating simultaneously the characteristic equations (Eq. 22) to obtain $v(z,\ell;\ z',v',\ell')$ and $\ell(z,\ell;z',v',\ell')$ for various z',v', and ℓ' and substituting these relationships into Eq. (23). Since $\alpha/\ell^2 >> 1$, it was very time consuming to integrate Eq. (22a) numerically. Referring to Fig. 1, it is seen that an appropriate approximation is to equate the velocity and the terminal velocity. This permits semi-analytical solutions.

This approximation can be verified by rewriting Eq. (22a) as [Ref. 5]

$$\frac{dv}{dz} = -\left(\frac{\xi}{\ell}\right)(v - v_T)(v + v'_T) \qquad (25)$$

where

$$v_T = \frac{-1 + [1 + 4(v_{T1}/v_{T2})^2]^{1/2}}{2(v_{T1}/v_{T2}^2)} \qquad (26)$$

and

$$v_T' = \frac{1 + [1 + 4(v_{T1}/v_{T2})^2]^{1/2}}{2(v_{T1}/v_{T2}^2)} \; . \qquad (27)$$

$$v_{T1} = \left[\frac{g}{\beta}\,\frac{\ell^2}{\alpha}\right] \qquad (28a)$$

is the terminal velocity in the case where viscous drag dominates ($\ell \leq 100\ \mu$).

$$v_{T2} = \left[\frac{g}{\beta}\,\frac{\ell}{\xi}\right]^{1/2} \qquad (28b)$$

is the terminal velocity in the case where turbulent drag dominates ($\ell \geq 400\ \mu$). Thus

$$v_T = \begin{cases} v_{T1} & \text{for} \quad \ell \leq 100 \\ v_{T2} & \text{for} \quad \ell \geq 400 \end{cases} \; . \qquad (29)$$

Equation (25) was integrated using fourth rank optional Runge Kutta [Ref. 11]. These numerical results are presented graphically in Fig. 1. It is observed that when a bubble is released from a depth of 20.00 meters with zero velocity, it will reach terminal velocity after having traveled a distance of only several bubble radii. Thus the approximation that $v = v_T$ is considered to be a valid approximation for this model. This approximation is convenient as Eq. (22a) is very difficult to integrate numerically.

By using the approximation $v = v_T$, the bubble distribution function can be written approximately as:

$$\phi(z,v,\ell) = \delta(v - v_T)\Phi(z,\ell) \quad (30)$$

where

$$\Phi(z,\ell) = \int_{-\infty}^{\infty} \phi(z,v,\ell)dv \quad (31)$$

is the "depth-radius" distribution function. By substituting Eq. (31) into Eq. (24) and integrating over v, an expression for Φ was determined as [Ref. 5]:

$$\phi(z,\ell) = \frac{v_{To}}{v_T}\,\phi(z_o,\ell_o)\left(\frac{\partial\ell_o}{\partial\ell}\right) + \frac{1}{v_T}\int_{z_o}^{z} s(z',\ell')\left(\frac{\partial\ell'}{\partial\ell}\right) dz' \quad (32)$$

where

$$s(z',\ell') \equiv \int_{-\infty}^{\infty} S(z',v',\ell')dv' \quad (33a)$$

and

$$d\ell_o = \left(\frac{\partial\ell_o}{\partial\ell}\right) d\ell \text{ , etc.} \quad (33b)$$

In order to calculate $\phi(z,\ell)$, it was necessary to determine $\partial\ell'/\partial\ell$ and $\ell'(z';z,\ell)$ either analytically or numerically. Note that $\partial\ell'/\partial\ell = 1/(\partial\ell/\partial\ell')$, and that the approximate differential equation that $\partial\ell/\partial\ell'$ satisfies is obtained by setting $v = v_T$ in Eq. (22b) and taking the partial derivative with respect to ℓ'[Ref. 5]:

$$\frac{\partial}{\partial z}\;\frac{\partial\ell}{\partial\ell'} = \left\{\left(\frac{2\Gamma}{\ell^2}\right)\left[\ell - \left(\frac{3\delta RT}{v_T}\right)\left(\frac{\Gamma}{\ell} - z\right)\right] \Big/ \left[3(D-z) + \frac{2\Gamma}{\ell}\right]\right.$$
$$\left. + \left[1 + \left(\frac{3\delta RT}{v_T}\right)\left[\frac{\Gamma}{\ell^2} + \left(\frac{\Gamma}{\ell} - z\right)\left(\frac{\partial v_T}{\partial\ell}\right)\Big/ v_T\right]\right]\right\} \Big/ \left[3(D-z) + \frac{2\Gamma}{\ell}\right] \quad (34a)$$

where

$$\frac{\partial v_T}{\partial\ell} = \frac{3v_{T1}}{\ell}\left[1 + 4\left(\frac{v_{T1}}{v_{T2}}\right)^2\right]^{-1/2} - \frac{v_T}{\ell} \, . \quad (34b)$$

Furthermore, with the approximation $v = v_T$, then $\ell = \ell_T$, and Eq. (22b)

becomes

$$\frac{d\ell_T}{dz} = \frac{\ell^2 + 3\delta RT(z\ell - \Gamma)/v_T}{3(D-z)\ell + 2\Gamma} \qquad (35)$$

Equations (34) and (35) can be integrated numerically using optimal fourth rank Runge Kutta [Ref. 11]. These results can be substituted into Eq. (32) to obtain $\Phi = \Phi_T$ for various sources. Since $v = v_T$ is a good approximation, it is expected that the approximation $\ell = \ell_T$ will also be very good.

To obtain approximate analytical results for comparison to the more exact numerical results outlined above, the approximations of Eq. (29) can be used. Neglecting surface tension, $\Gamma = 0$ is also a valid approximation for $\ell \geq 30\ \mu$. Then Eq. (35) can be integrated analytically [Ref. 5]. For $v = v_{T1}$ and $\Gamma = 0$

$$\ell = \ell_{T1}(z;z',\ell') = \left[\frac{D-z}{D-z}\,\ell'^3 + \frac{3}{2}\,\frac{\alpha\delta RT}{(g/\beta)}\;\frac{z^2\ \ z'^2}{D-z}\right]^{1/3} . \qquad (36a)$$

For $v = v_{T2}$ and $\Gamma = 0$

$$\ell = \ell_{T2}(z,z',\ell') = \left\{\left(\frac{D-z'}{D-z}\right)^{1/2} \ell'^{3/2} + \frac{\delta RT}{((g/\beta)\xi)^{1/2}}\left[(D-z)\left(1-\left(\frac{D-z'}{D-z}\right)^{3/2}\right) - 3D\left(1-\left(\frac{D-z'}{D-z}\right)^{1/2}\right)\right]\right\}^{2/3} . \qquad (37a)$$

The inverse transformation are

$$\ell = \ell_{T1}(z';z,\ell) \qquad (36b)$$

and

$$\ell = \ell_{T2}(z';z,\ell) \qquad (37b)$$

respectively. Then

$$\frac{\partial \ell'}{\partial \ell} = \left(\frac{D-z}{D-z'}\right)\frac{\ell^2}{\ell'^2} \qquad (36c)$$

and

$$\frac{\partial \ell'}{\partial \ell} = \left(\frac{D-z}{D-z'}\right)^{1/2} \left(\frac{\ell}{\ell'}\right)^{1/2} \tag{37c}$$

respectively. If gas diffusion can be neglected, $\delta = 0$, then Eqs. (36) and (37) reduce to:

$$\frac{\ell}{\ell'} = \left(\frac{D-z'}{D-z}\right)^{1/3} \tag{38a}$$

$$\frac{\partial \ell'}{\partial \ell} = \left(\frac{D-z}{D-z'}\right)^{1/3} . \tag{38b}$$

Figures (2), (3), (4), (5), and (6) demonstrate the results obtained by integrating Eq. (35) numerically and comparing ℓ_T to the analytical approximations of Eqs. (36), (37), and (38). Figures (2), (3), and (4) were drawn for a depth of 20 meters and Figs. (5) and (6) were drawn for a depth of 197 meters. Figures (2) and (5) demonstrate that ℓ_{T1} is a good approximation for ℓ_T when $\ell \leq 100$ μ. However, this approximation breaks down when $\ell < 30$ μ. This is because the effects of surface tension, which are important in bubbles of $\ell \leq 30$ μ, are neglected. Figures (3), (4), and (6) demonstrate that ℓ_{T2} is a good approximation for ℓ_T when $\ell \geq 400$ μ. From Fig. (6), it is observed that ℓ_{T3} is not a good approximation for ℓ. This is because ℓ_{T3} neglects the effects of gas diffusion.

The survival threshold for bubbles is different depending on whether the bubbles were released at a depth 20 meters or 197 meters. In the 20 meter case, bubbles of $\ell \geq 100$ μ will reach the surface, while in the 197 meter case, bubbles of $\ell \geq 1000$ meters will surface. This difference is caused by gas diffusion being dominate over the decrease in hydrostatic pressure. Since gas diffusion is proportional to $(P-P_o)/P$, it will be constant for $z \leq 30$ meters and will only

decrease for $z \geq 30$ meters. Thus bubbles released at 197 meters are exposed to a high gas diffusion rate for practically all their travel, while bubbles released at 20 meters are exposed to a decreasing gas diffusion rate.

Finally the bubble distribution function was calculated for each of these approximate solutions to Eqs. (22a) and (22b), (34) and (35), (36), (37), and (38). It was convenient to introduce the initial distribution function as the boundary condition $\Phi(z;z_o,\ell_o)$ instead of as a distributed source. Since $s(z',\ell')$ is zero, the solution can be written as

$$\Phi(z;z_o,\ell_o) = \frac{V_{To}}{V_T} \; \Phi(z_o,\ell_o) \; 1/(\partial\ell/\partial\ell'). \tag{39a}$$

For $v = v_T$, Eqs. (34) and (35) must be integrated numerically to yield $\Phi_T(z';z_o,\ell_o)$. For $v = v_{T1}(\ell \leq 100\mu)$ and $\Gamma = 0$

$$\Phi \approx \Phi_1(z;z_o,\ell_o) = \left(\frac{D-z}{D-z_o}\right) \Phi(z_o,\ell_o); \tag{39b}$$

for $v = v_{T2}(\ell \geq 400\mu)$

$$\Phi \approx \Phi_2(z;z_o,\ell_o) = \left(\frac{D-z}{D-z_o}\right)^{1/2} \Phi\,(z_o,\ell_o); \tag{39c}$$

and if the effects of gas diffusion are not considered

$$\Phi \approx \Phi_3(z;z_o,\ell_o) = \frac{V_{To}}{V_T} \Phi(z_o,\ell_o) \left(\frac{D-z}{D-z_o}\right)^{1/3} \tag{39d}$$

Here V_{T3} is computed using

$$\ell = \ell_o \left(\frac{D-z}{D-z_o}\right)^{1/3}$$

in Eq. (26).

The above distributions were calculated using an initial distribution function, $\Phi(z_o,\ell_o)$, obtained from Ref. 4 . The authors of Ref. 4

determined the distribution of bubbles rising from "the highly organic, anaerobic sediment on the bottom" of Saanich Inlet, British Columbia. The depth of the water was 197 meters. The density of these rising bubbles was determined by using stationary echo-sounders operating at various frequencies in the kilohertz range. Figure 7a is a histogram that presents the presumed bubble distribution, $\Phi(z_o,\ell_o)$ at a depth of 197 meters as presented in Ref. 4. Figures (7b) through (7g) demonstrate how this initial distribution changes as the bubbles rise in height. It should be noted that height is measured from the ocean bottom upward to the surface.

The different histograms at each height above the ocean floor shows how the bubble distribution is changing depending on the model chosen to be used for the calculation of ℓ_T. In the model where gas diffusion as well as hydrostatic pressure changes are considered, the bubble distribution $\Phi(z;z_o,\ell_o)$ shifts toward smaller size bubbles as height increases. It can also be observed that the width of the intervals is increasing with height. Finally all the bubbles disappear entirely at a height of about 100 meters.

In the small bubble approximation $\Phi \approx \Phi_1(z;z_o,\ell_o)$, where the drag force is considered to be viscous and $v = v_{T1}$, the bubble distribution appears to spread out by shifting toward bubbles of both smaller and larger radii. Bubbles with initial radius less than 900 μ continually decrease in radius, while bubbles with an initial radius greater than 900 μ will initially decrease in size but as the effects of gas diffusion reduce, these bubbles expand. In the turbulent drag model, $v = v_{T2}$, $\Phi = \Phi_2(z;z_o,\ell_o)$ shifts toward smaller radius bubbles because of the effect of gas diffusion and disappears at a height of about 125 meters.

When gas diffusion is neglected, the bubble distribution $\Phi \approx \Phi_3(z;z_o,\ell_o)$ shifts toward bubbles of larger radii. This was to be expected because as hydrostatic pressure decreased, the bubbles would increase in size. The bubble distribution calculated for $\Phi_3(z;z_o,\ell_o)$ compares quite well with the experimental data of McCartney and Bary in Ref. 4. It appears as though gas diffusion was somehow not important in the bubbles they measured. These bubbles originated at the bottom, so perhaps they had some kind of coating that impeded gas diffusion across their surfaces.

VI. CONCLUSION

Calculating bubble distributions in the ocean by solving the bubble transport equation appeared to be quite feasible. Although the solution of the transport equation can be quite involved, a careful choice of approximations often permits approximate analytical solutions. Otherwise, the equations can usually be numerically integrated. For the model chosen, analytical expressions were obtained for several approximations. All of these expressions yielded results that were in general agreement with experimental data or with intuitive reasoning. It was seen that the drag force model chosen and the inclusion of gas diffusion greatly influenced the results obtained.

The assumption that the velocity of the bubble was equal to the terminal velocity should be quite valid as the data presented on Fig. 1 demonstrated. With this approximation, one can easily calculate the bubble radius at any depth. The radius is strongly dependent on the model chosen for the drag force, gas diffusion and surface tension. The choice of the drag model determines the equation used to calculate the terminal velocity.

Finally the distribution function for a bottom source was calculated. The data generated by this theoretical approach agrees well with the limited experimental data available. The distribution calculated for the no gas diffusion model yields data that is in good agreement with the experimental data in Ref. 4.

From the limited amount of experimental data available, it appears that gas diffusion is unimportant in bubbles originating in bottom

sediment and transported to the surface. It is felt that the type of gas in the bubbles would greatly influence the amount of diffusion that would take place [Ref. 12]. Since a fluid, super-saturated with the same gas that is within the bubble, is the only fluid in which gas diffusion is not considered the dominating effect, there must be some mechanism inhibiting gas diffusion from sediment bubbles. Further investigation into this phenomena seems warranted.

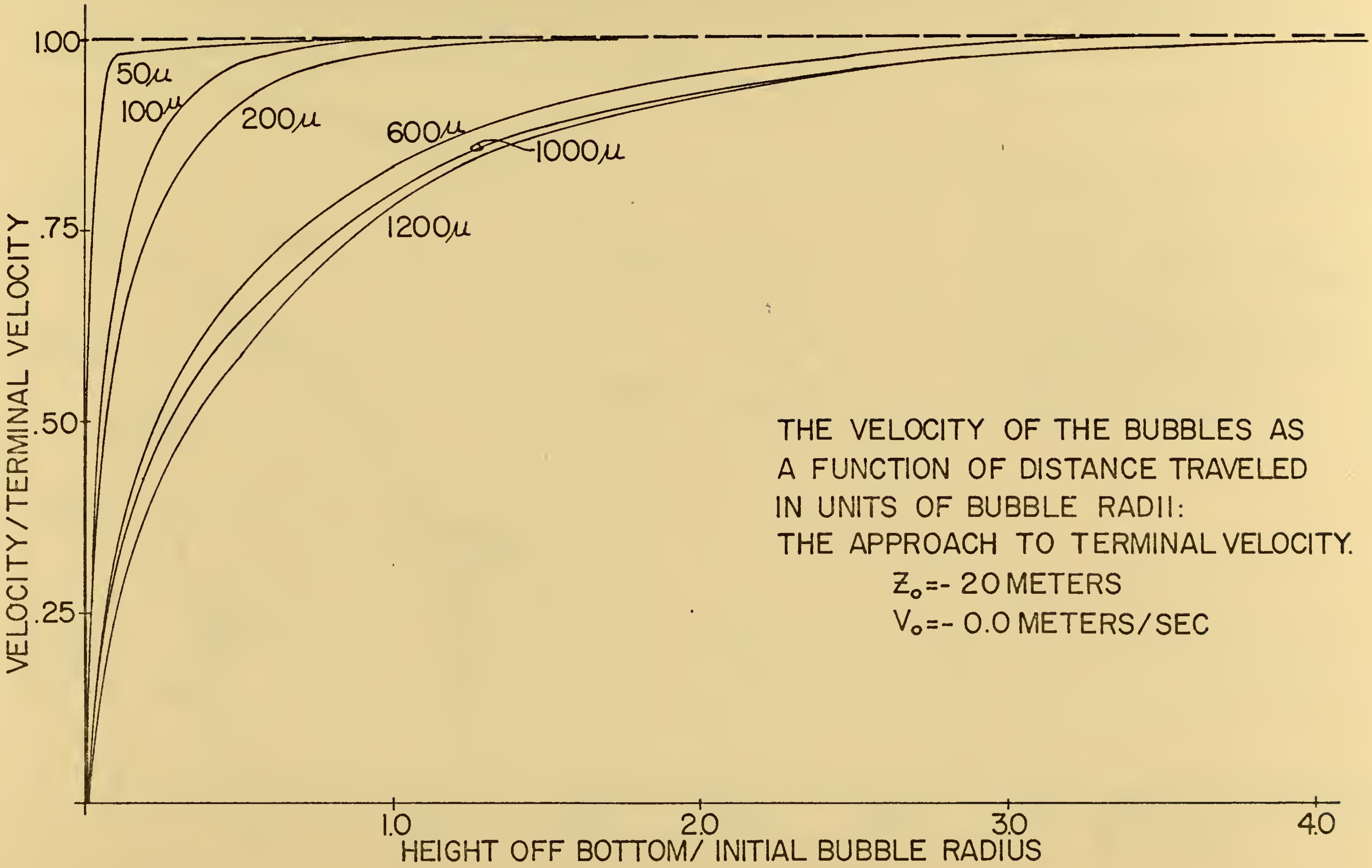
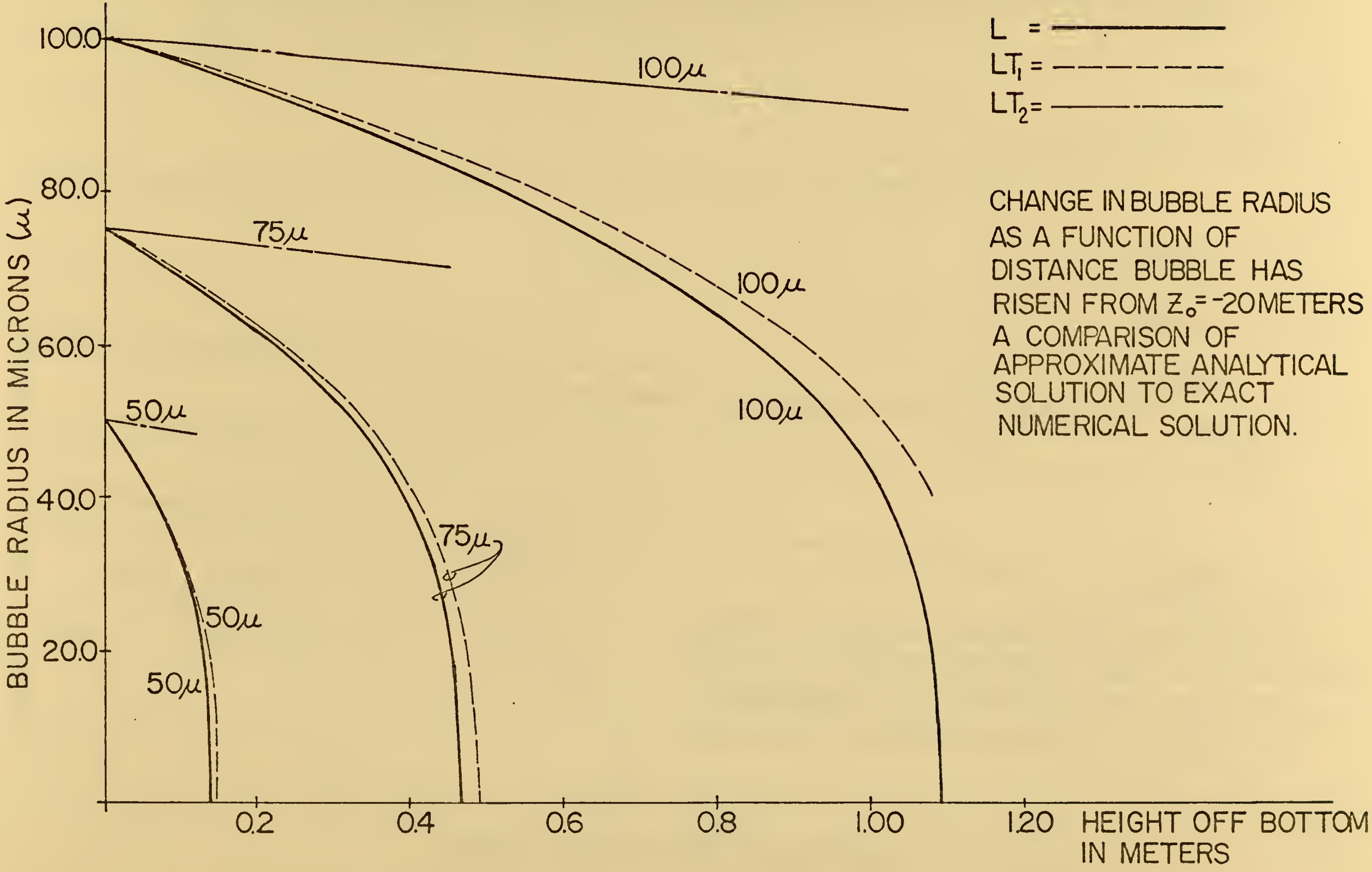
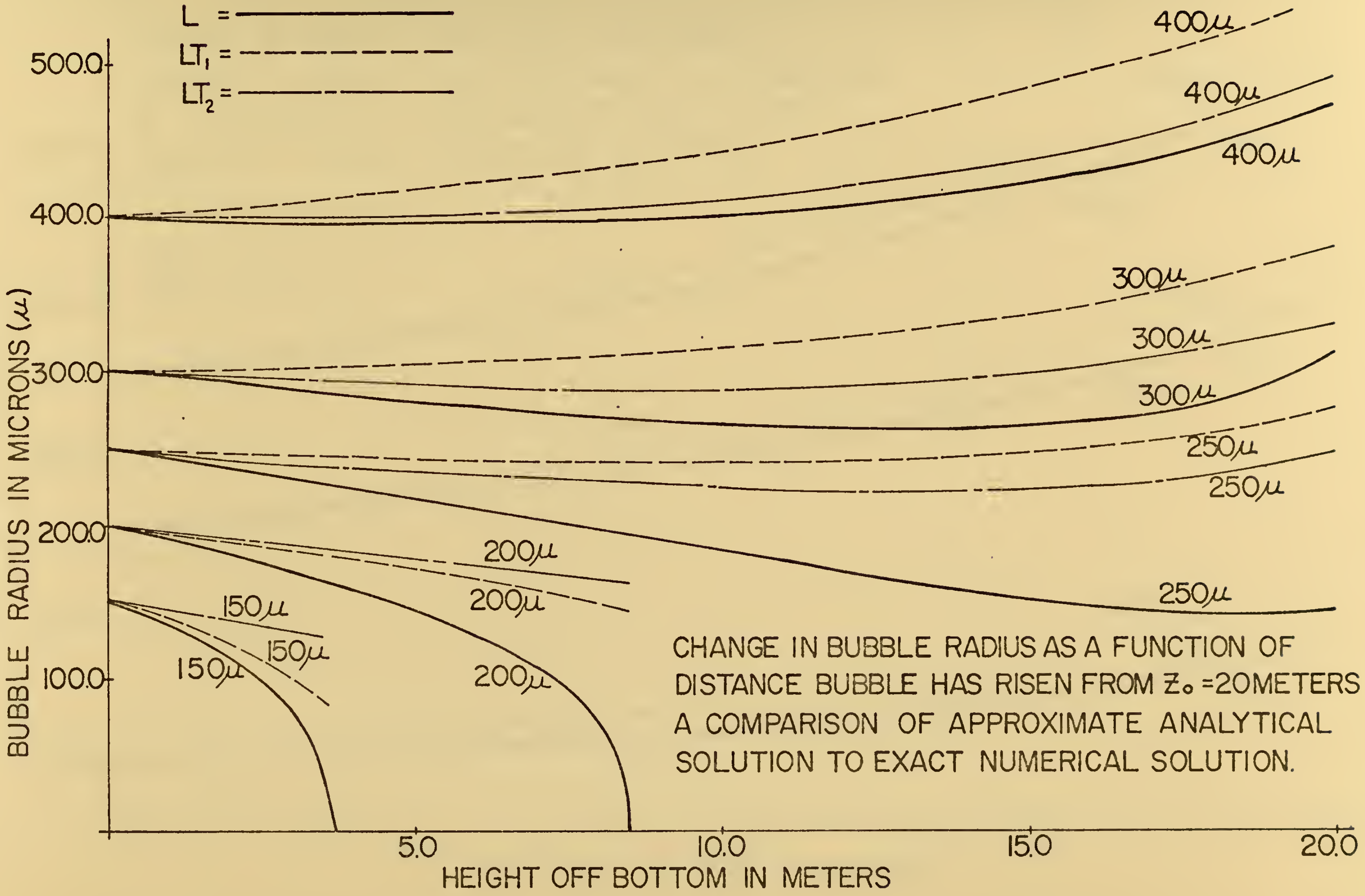

THE VELOCITY OF THE BUBBLES AS
A FUNCTION OF DISTANCE TRAVELED
IN UNITS OF BUBBLE RADII:
THE APPROACH TO TERMINAL VELOCITY.
Z_o=- 2.0 METERS
V_o=- 0.0 METERS/SEC

Figure 1



CHANGE IN BUBBLE RADIUS AS A FUNCTION OF DISTANCE BUBBLE HAS RISEN FROM $Z_o = -20$ METERS A COMPARISON OF APPROXIMATE ANALYTICAL SOLUTION TO EXACT NUMERICAL SOLUTION.

Figure 2



CHANGE IN BUBBLE RADIUS AS A FUNCTION OF DISTANCE BUBBLE HAS RISEN FROM Z_o =20METERS A COMPARISON OF APPROXIMATE ANALYTICAL SOLUTION TO EXACT NUMERICAL SOLUTION.

Figure 3

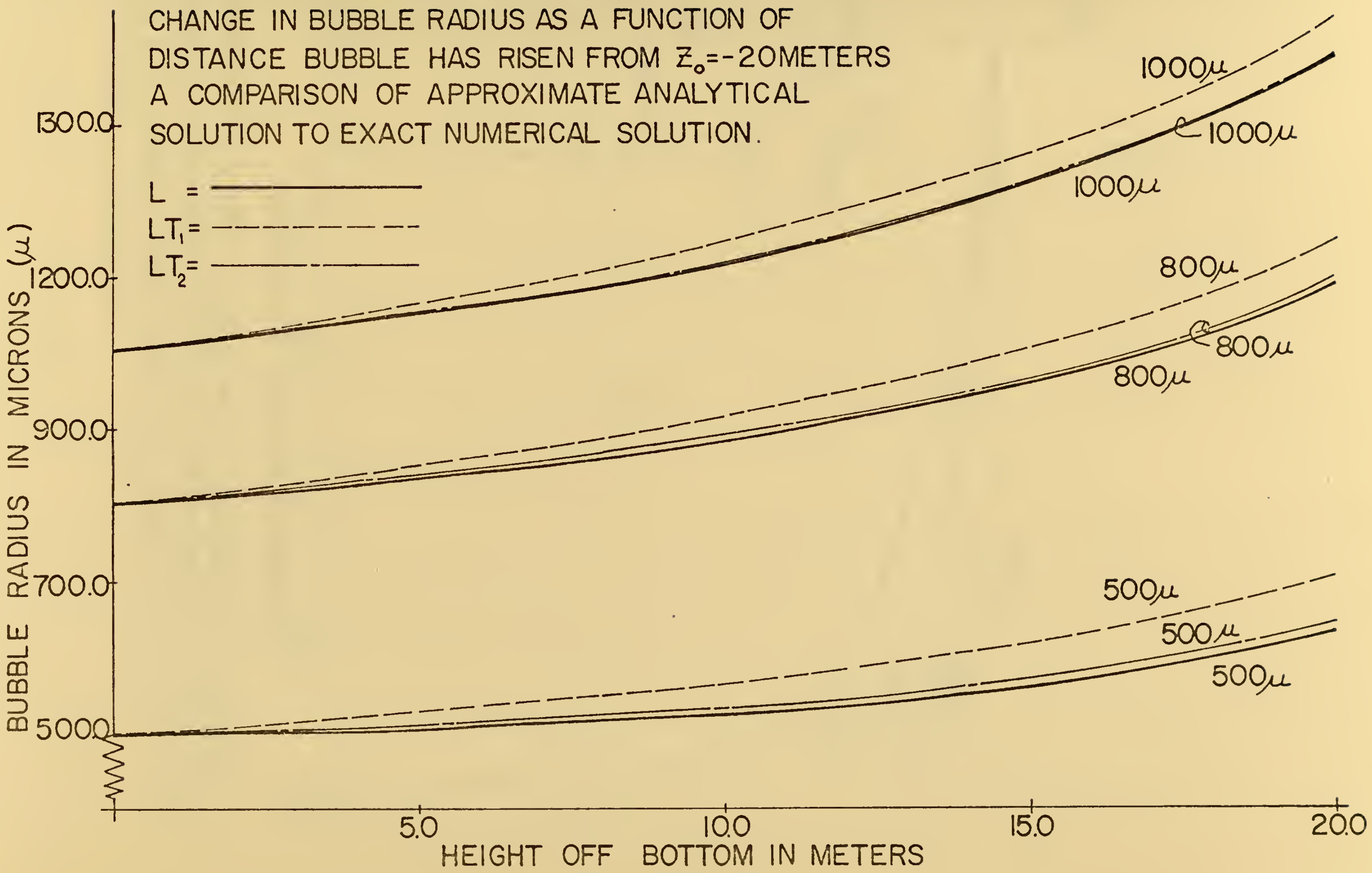
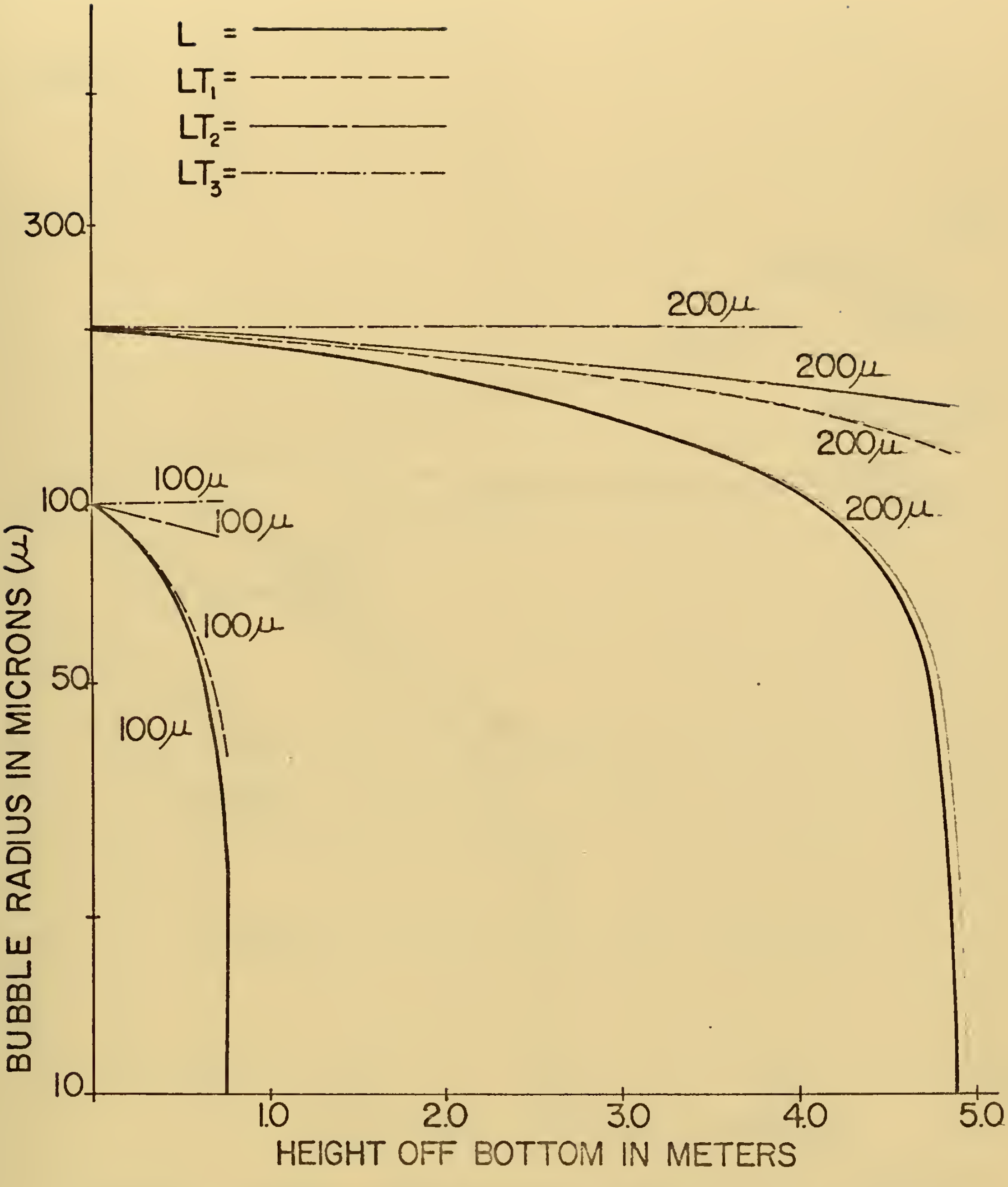
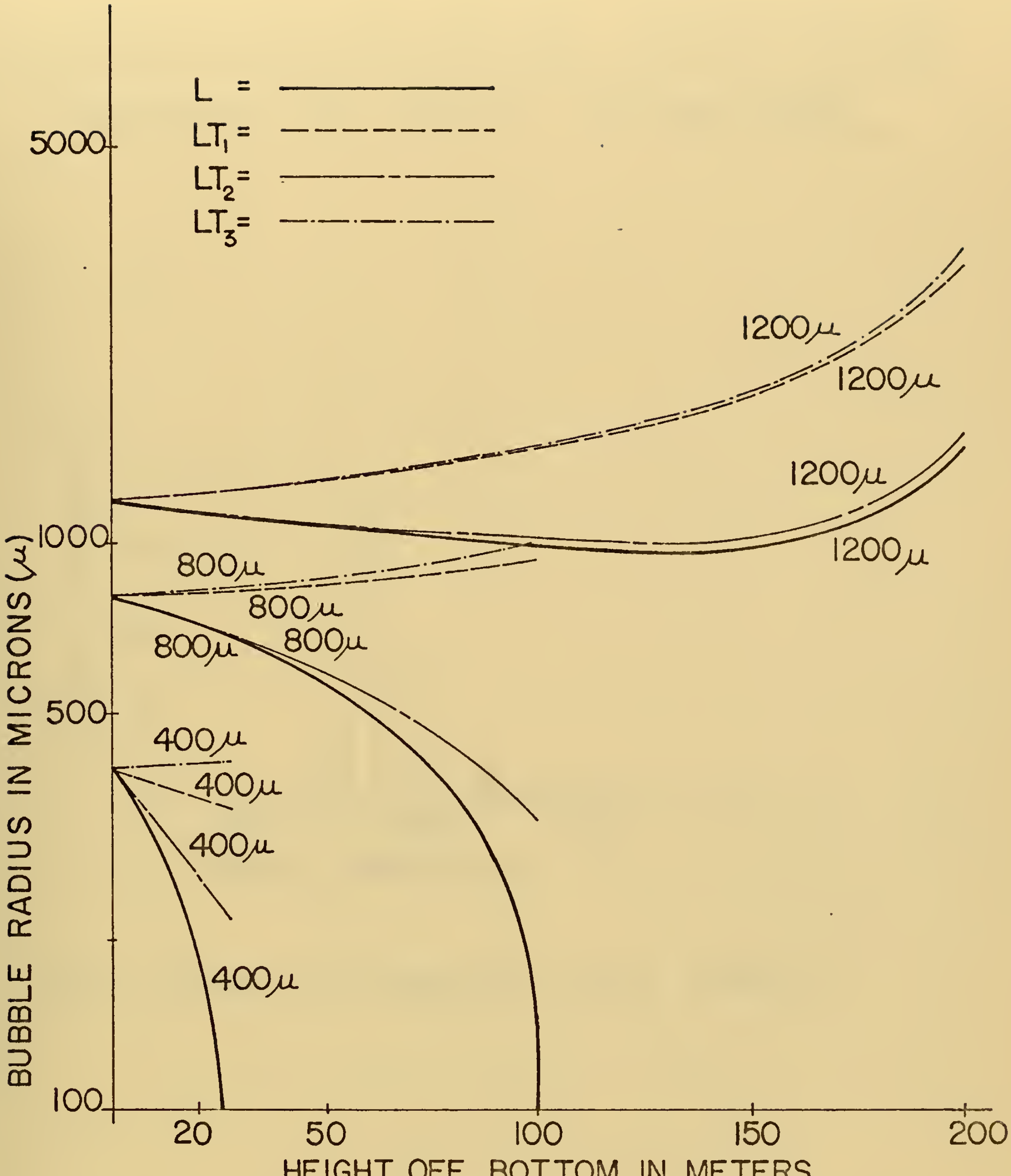

CHANGE IN BUBBLE RADIUS AS A FUNCTION OF
DISTANCE BUBBLE HAS RISEN FROM Z_o=-20METERS
A COMPARISON OF APPROXIMATE ANALYTICAL
SOLUTION TO EXACT NUMERICAL SOLUTION.
L =
LT$_1$=
LT$_2$=
1000μ
1000μ
1000μ
800μ
800μ
800μ
500μ
500μ
500μ
1300.0
1200.0
900.0
700.0
500.0
BUBBLE RADIUS IN MICRONS (μ)
5.0
10.0
15.0
20.0
HEIGHT OFF BOTTOM IN METERS

Figure 4



CHANGE IN BUBBLE RADIUS AS A FUNCTION OF DISTANCE BUBBLE HAS RISEN FROM $Z_0 = -197$ METERS A COMPARISON OF APPROXIMATE ANALYTICAL SOLUTION TO EXACT NUMERICAL SOLUTION.

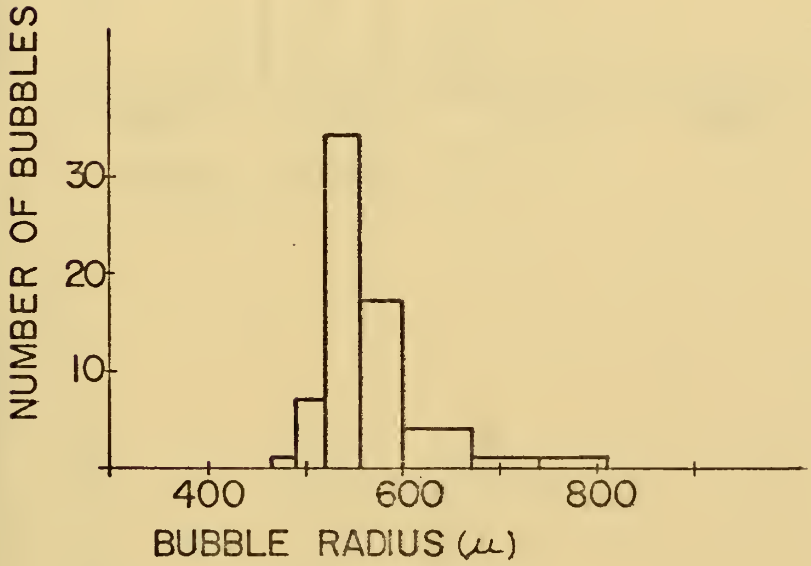
Figure 5



CHANGE IN BUBBLE RADIUS AS A FUNCTION OF DISTANCE BUBBLE HAS RISEN FROM Z_o =-197 METERS A COMPARISON OF APPROXIMATE ANALYTICAL SOLUTION TO EXACT NUMERICAL SOLUTION.

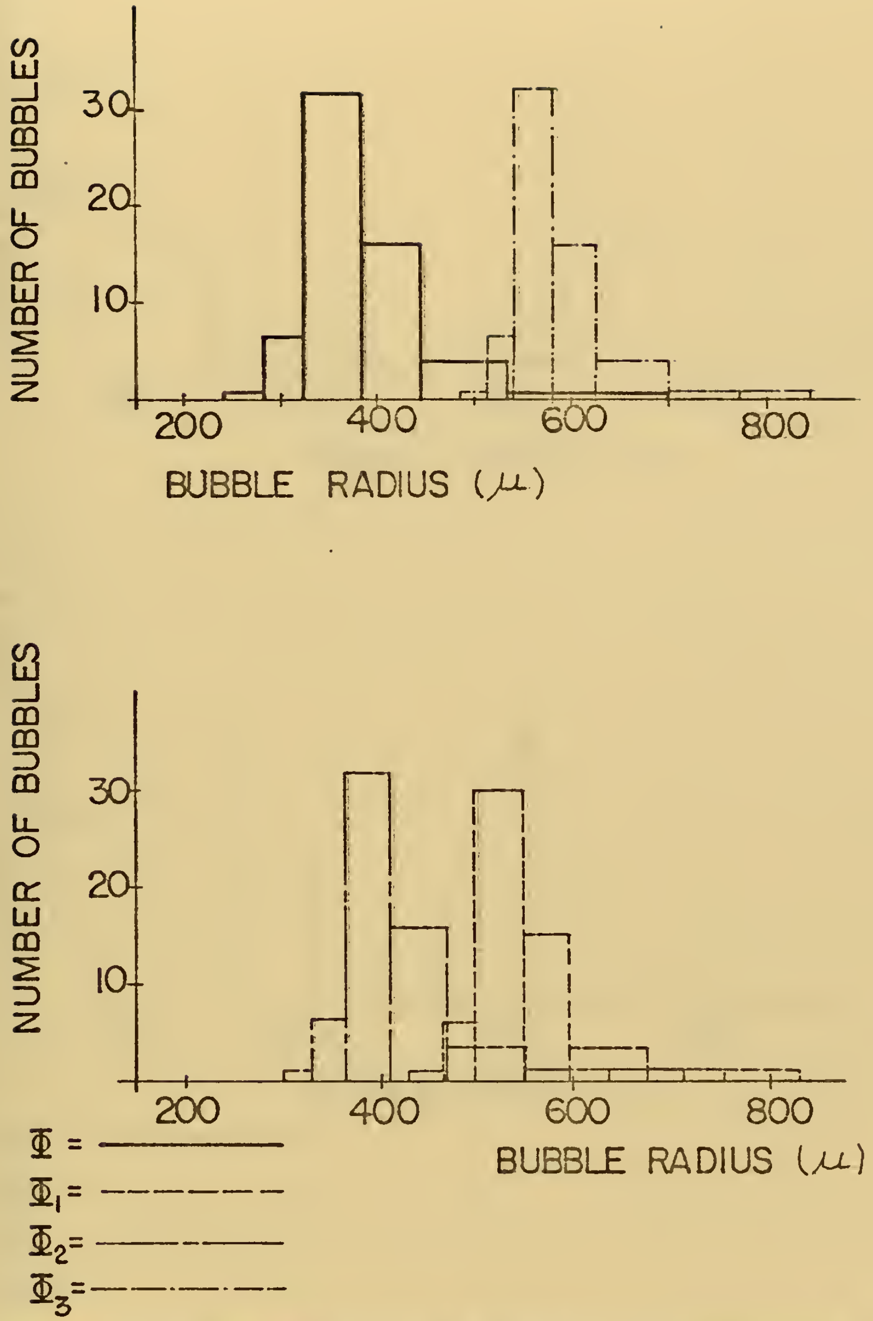
Figure 6

DISTRIBUTION OF BUBBLES ORIGINATING IN SEDIMENT AT DEPTH 197 METERS.



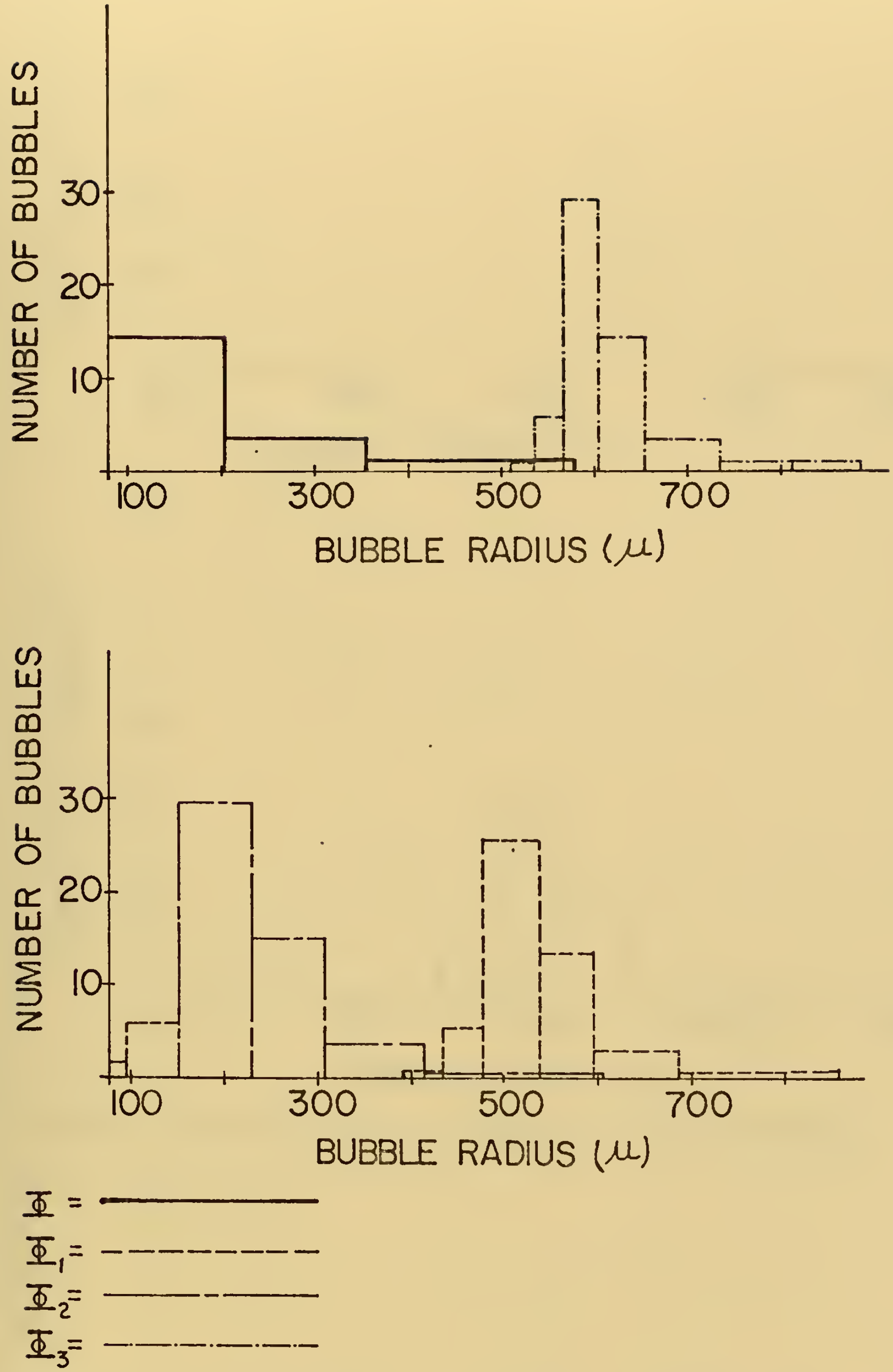
INITIAL BUBBLE DISTRIBUTION $\Phi(Z,\ell)$ (REF. 4)

Figure 7a



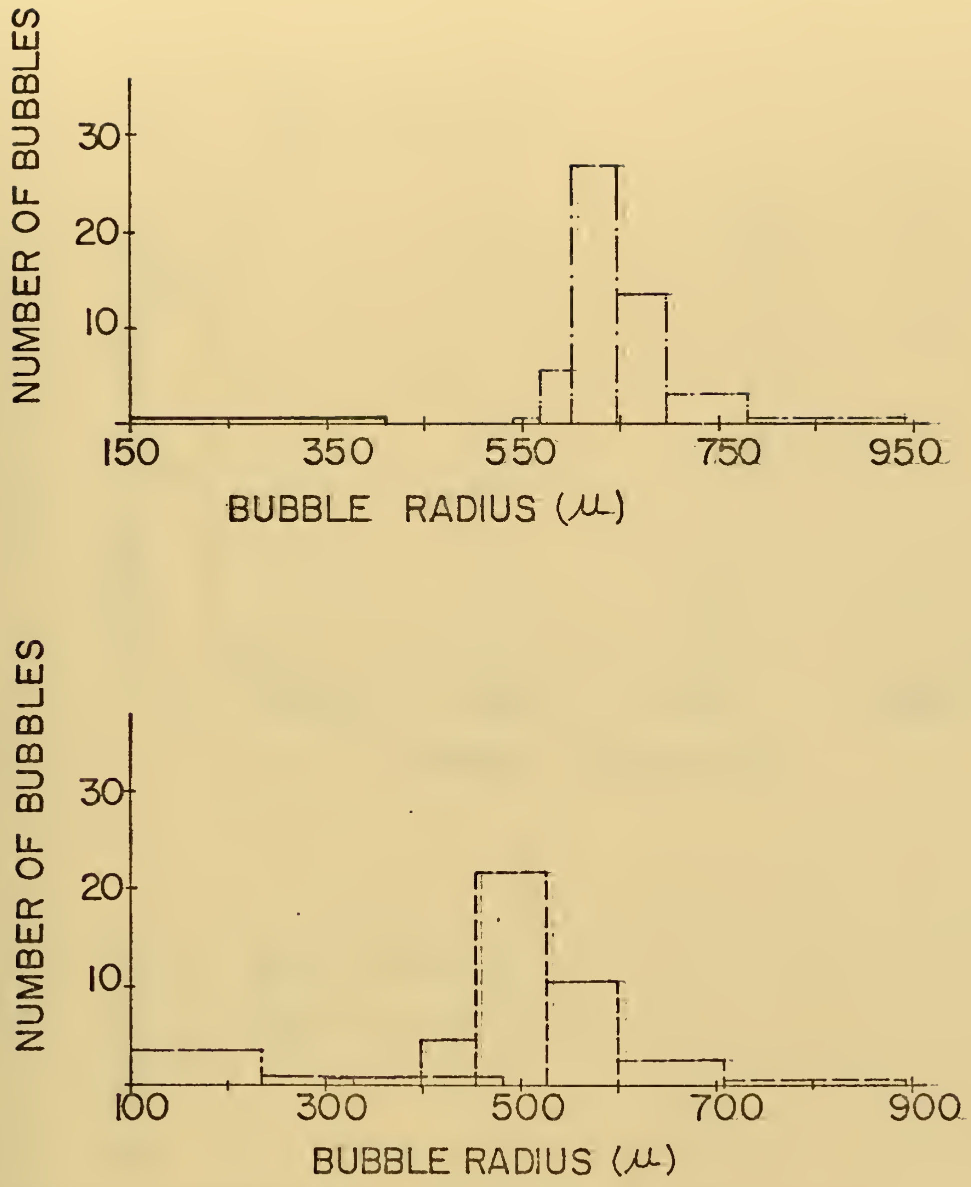
BUBBLE DISTRIBUTION AT HEIGHT = 25 METERS.

Figure 7b



BUBBLE DISTRIBUTION AT HEIGHT = 50 METERS.

Figure 7c



BUBBLE DISTRIBUTION AT HEIGHT = 75 METERS.

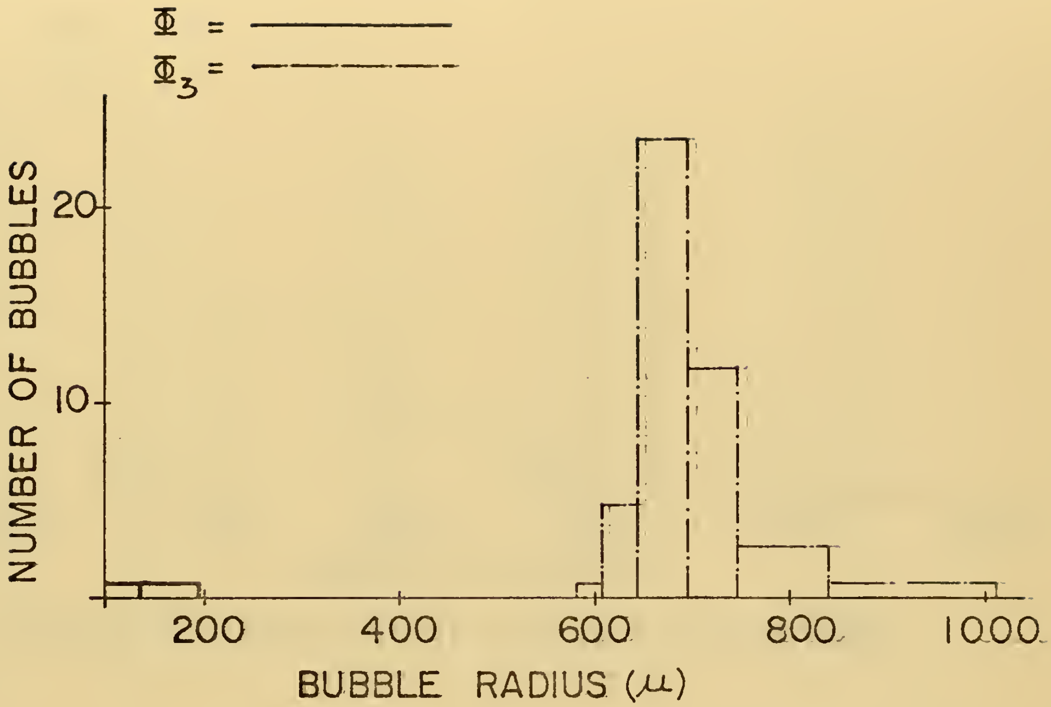
Φ = ————————

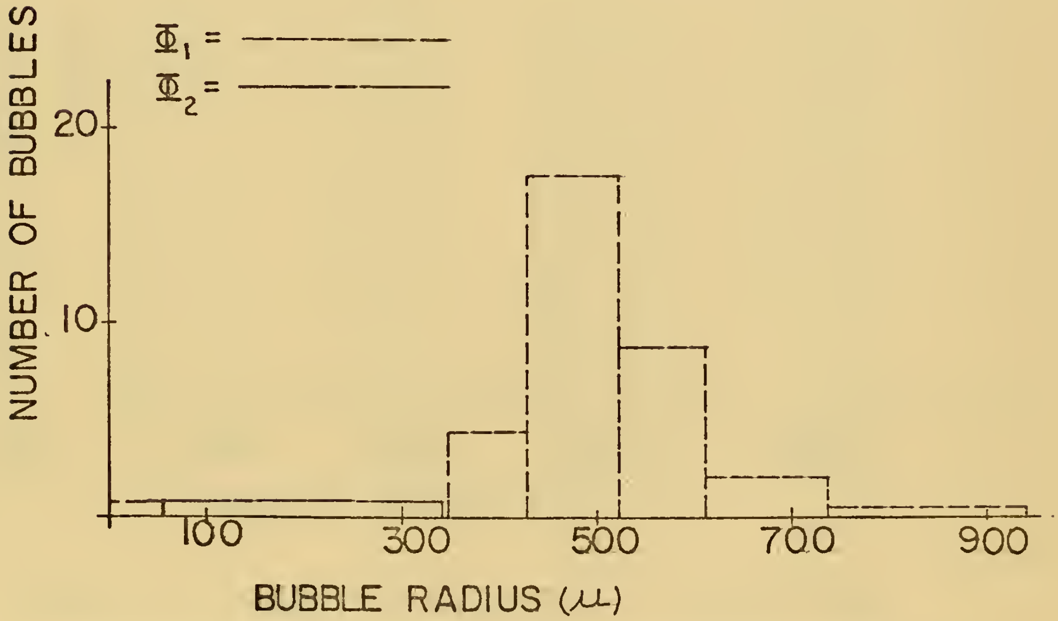
Φ_1 = — — — — — — —

Φ_2 = —————— - ———

Φ_3 = ——·——·——·——

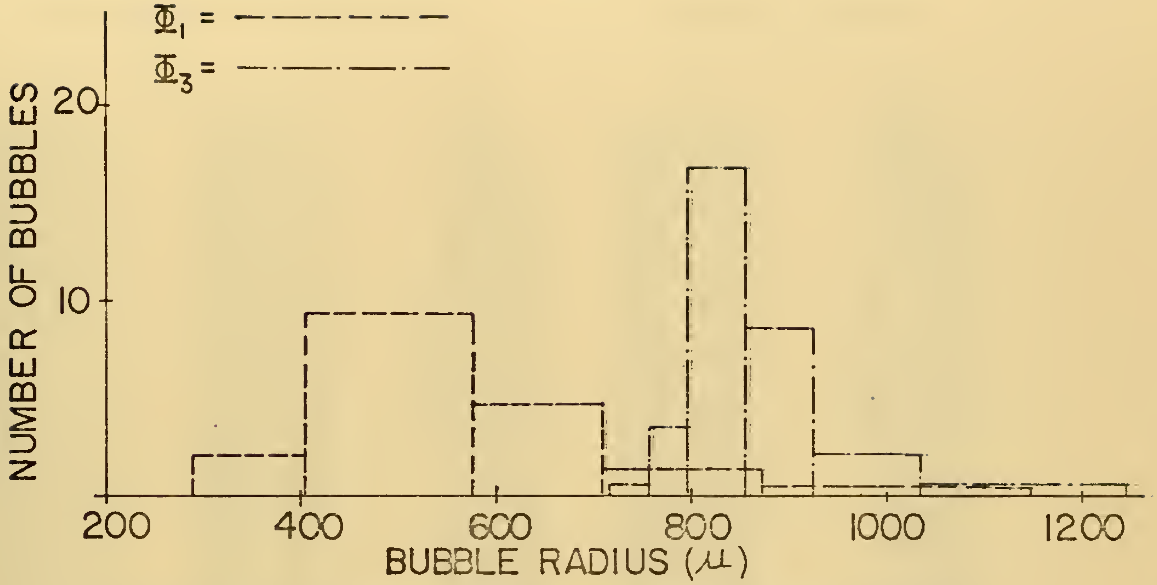
Figure 7d





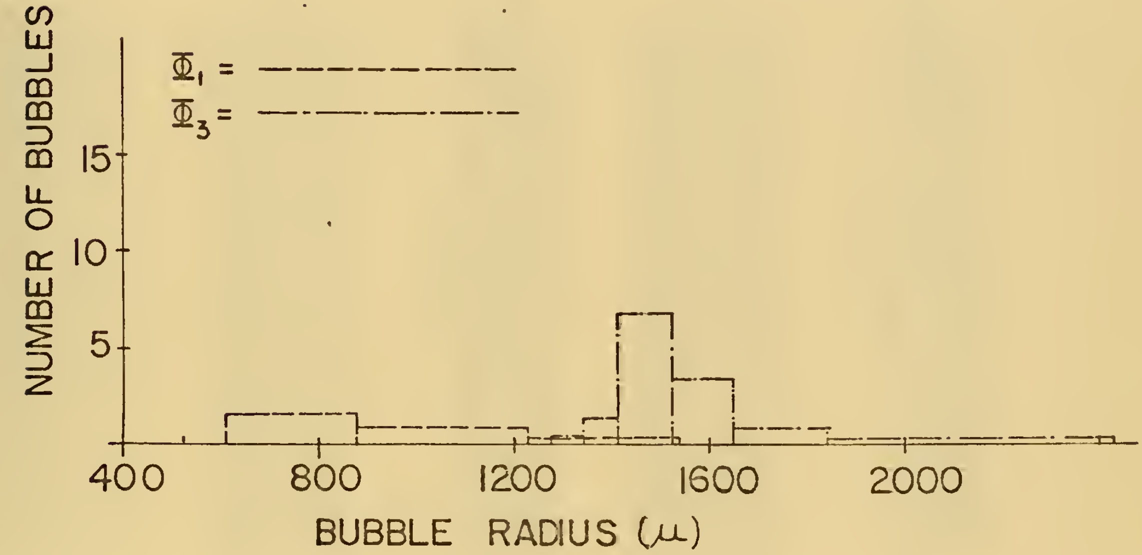
BUBBLE DISTRIBUTION AT HEIGHT = 100 METERS.

Figure 7e



BUBBLE DISTRIBUTION AT HEIGHT = 150 METERS.

Figure 7f



BUBBLE DISTRIBUTION AT HEIGHT = 197 METERS.

Figure 7g

COMPUTER OUTPUT

```
HEIGHT OFF THE BOTTOM =        25.000 METERS


   XLO          RADIUS        PHI          RLT3          PHI3
400.000          0.0          0.0        417.535        0.0
465.000        240.062        0.924      485.385        0.929
475.000        258.600        0.925      495.823        0.929
485.000        276.132        0.926      506.261        0.930
490.000        284.593        6.482      511.480        6.509
500.000        301.010        6.486      521.919        6.510
510.000        316.856        6.490      532.357        6.512
515.000        324.596       31.530      537.576       31.633
530.000        347.201       31.552      553.234       31.643
545.000        369.032       31.572      568.892       31.653
555.000        383.230       15.792      579.330       15.830
575.000        410.924       15.802      600.207       15.835
595.000        437.842       15.812      621.083       15.840
600.000        444.469        3.721      626.303        3.727
635.000        489.908        3.724      662.837        3.729
660.000        521.523        3.726      688.933        3.730
670.000        534.007        0.932      699.371        0.933
705.000        577.089        0.932      735.906        0.933
730.000        607.364        0.933      762.002        0.933
740.000        619.375        0.933      772.440        0.933
775.000        661.022        0.933      808.974        0.934
800.000        690.444        0.933      835.070        0.934
810.000        702.146        0.0        845.509        0.0



   XLO           RLT1         PHI1          RLT2          PHI2
400.000        338.683        0.0        206.200        0.0
465.000        431.628        0.879      297.174        0.938
475.000        444.714        0.879      310.439        0.938
485.000        457.588        0.879      323.562        0.938
490.000        463.954        6.155      330.075        6.564
500.000        476.556        6.155      343.009        6.564
510.000        489.000        6.155      355.830        6.564
515.000        495.169       29.894      362.201       31.881
530.000        513.480       29.894      381.168       31.881
545.000        531.533       29.894      399.935       31.881
555.000        543.442       14.947      412.346       15.940
575.000        566.997       14.947      436.950       15.940
595.000        590.247       14.947      461.293       15.940
600.000        596.018        3.517      467.342        3.751
635.000        636.015        3.517      509.318        3.751
660.000        664.219        3.517      538.952        3.751
670.000        675.430        0.879      550.735        0.938
705.000        714.391        0.879      591.686        0.938
730.000        741.996        0.879      620.693        0.938
740.000        752.992        0.879      632.244        0.938
775.000        791.303        0.879      672.467        0.938
800.000        818.520        0.879      701.018        0.938
810.000        829.376        0.0        712.401        0.0
```

```
HEIGHT OFF THE BOTTOM =        50.000 METERS


   XLO           RADIUS        PHI          RLT3          PHI3
400.000          0.0         0.0          438.615        0.0
465.000          0.0         0.0          509.890        0.855
475.000          0.0         0.0          520.855        0.855
485.000          0.0         0.0          531.821        0.856
490.000          0.0         0.0          537.303        5.991
500.000          0.0         0.0          548.269        5.994
510.000          0.0         0.0          559.234        5.997
515.000          0.0         0.0          564.717       29.136
530.000          0.0         0.0          581.165       29.155
545.000          0.0         0.0          597.613       29.174
555.000          0.0         0.0          608.578       14.593
575.000         92.385      14.316        630.509       14.603
595.000        188.945      14.403        652.440       14.613
600.000        204.448       3.393        657.923        3.439
635.000        290.264       3.410        696.301        3.443
660.000        340.102       3.419        723.715        3.445
670.000        358.626       0.855        734.680        0.861
705.000        419.267       0.857        773.059        0.862
730.000        459.715       0.858        800.473        0.863
740.000        475.396       0.859        811.438        0.863
775.000        528.524       0.860        849.817        0.863
800.000        565.120       0.861        877.230        0.864
810.000        579.499       0.0          888.196        0.0



   XLO            RLT1         PHI1         RLT2          PHI2
400.000        222.533       0.0          0.0            0.0
465.000        389.746       0.758        0.0            0.0
475.000        408.047       0.758       48.238          0.871
485.000        425.532       0.758       79.764          0.871
490.000        434.014       5.309       93.391          6.096
500.000        450.529       5.309      118.240          6.096
510.000        466.521       5.309      140.916          6.096
515.000        474.345      25.787      151.662         29.610
530.000        497.222      25.787      182.147         29.610
545.000        519.336      25.787      210.647         29.610
555.000        533.722      12.894      228.815         14.805
575.000        561.775      12.894      263.610         14.805
595.000        589.029      12.894      296.797         14.805
600.000        595.736       3.034      304.888          3.484
635.000        641.693       3.034      359.687          3.484
660.000        673.641       3.034      397.238          3.484
670.000        686.252       0.758      411.959          0.871
705.000        729.760       0.758      462.363          0.871
730.000        760.329       0.758      497.464          0.871
740.000        772.455       0.758      511.326          0.871
775.000        814.509       0.758      559.139          0.871
800.000        844.226       0.758      592.707          0.871
810.000        856.048       0.0        606.015          0.0
```

```
    HEIGHT OFF THE BOTTOM =         75.000 METERS


  XLO          RADIUS        PHI          RLT3         PHI3
400.000         0.0         0.0         464.721       0.0
465.000         0.0         0.0         540.238       0.775
475.000         0.0         0.0         551.856       0.776
485.000         0.0         0.0         563.474       0.777
490.000         0.0         0.0         569.283       5.440
500.000         0.0         0.0         580.901       5.444
510.000         0.0         0.0         592.519       5.449
515.000         0.0         0.0         598.328      26.474
530.000         0.0         0.0         615.755      26.502
545.000         0.0         0.0         633.182      26.528
555.000         0.0         0.0         644.800      13.272
575.000         0.0         0.0         668.036      13.288
595.000         0.0         0.0         691.272      13.302
600.000         0.0         0.0         697.081       3.131
635.000         0.0         0.0         737.744       3.136
660.000         0.0         0.0         766.789       3.139
670.000         0.0         0.0         778.407       0.785
705.000       148.353       0.766       819.070       0.786
730.000       244.821       0.772       848.115       0.787
740.000       273.440       0.774       859.734       0.787
775.000       358.003       0.778       900.397       0.788
800.000       410.244       0.780       929.442       0.788
810.000       429.964       0.0         941.060       0.0



  XLO           RLT1        PHI1          RLT2         PHI2
400.000         0.0         0.0           0.0         0.0
465.000       331.106       0.638         0.0         0.0
475.000       360.093       0.638         0.0         0.0
485.000       386.043       0.638         0.0         0.0
490.000       398.163       4.464         0.0         0.0
500.000       421.059       4.464         0.0         0.0
510.000       442.507       4.464         0.0         0.0
515.000       452.784      21.681         0.0         0.0
530.000       482.162      21.681         0.0         0.0
545.000       509.784      21.681         0.0         0.0
555.000       527.421      10.841         0.0         0.0
575.000       561.206      10.841         0.0         0.0
595.000       593.406      10.841        12.645      13.575
600.000       601.252       2.551        42.242       3.194
635.000       654.335       2.551       153.845       3.194
660.000       690.674       2.551       212.749       3.194
670.000       704.917       0.638       234.349       0.799
705.000       753.684       0.638       304.416       0.799
730.000       787.663       0.638       350.758       0.799
740.000       801.088       0.638       368.659       0.799
775.000       847.439       0.638       429.073       0.799
800.000       880.028       0.638       470.501       0.799
810.000       892.959       0.0         486.742       0.0
```

```
      HEIGHT OFF THE BOTTOM =        100.000 METERS


     XLO          RADIUS        PHI          RLT3          PHI3
   400.000        0.0         0.0          498.412        0.0
   465.000        0.0         0.0          579.404        0.690
   475.000        0.0         0.0          591.865        0.691
   485.000        0.0         0.0          604.325        0.692
   490.000        0.0         0.0          610.555        4.846
   500.000        0.0         0.0          623.016        4.851
   510.000        0.0         0.0          635.476        4.856
   515.000        0.0         0.0          641.706       23.599
   530.000        0.0         0.0          660.396       23.634
   545.000        0.0         0.0          679.087       23.666
   555.000        0.0         0.0          691.547       11.843
   575.000        0.0         0.0          716.468       11.862
   595.000        0.0         0.0          741.388       11.880
   600.000        0.0         0.0          747.619        2.796
   635.000        0.0         0.0          791.230        2.803
   660.000        0.0         0.0          822.380        2.807
   670.000        0.0         0.0          834.841        0.702
   705.000        0.0         0.0          878.452        0.703
   730.000        0.0         0.0          909.603        0.704
   740.000        0.0         0.0          922.063        0.704
   775.000        0.0         0.0          965.674        0.705
   800.000      136.578       0.680        996.825        0.706
   810.000      194.535       0.0         1009.285        0.0



     XLO           RLT1         PHI1          RLT2          PHI2
   400.000        0.0         0.0          0.0           0.0
   465.000      219.089       0.517        0.0           0.0
   475.000      285.770       0.517        0.0           0.0
   485.000      332.348       0.517        0.0           0.0
   490.000      351.978       3.618        0.0           0.0
   500.000      386.703       3.618        0.0           0.0
   510.000      417.224       3.618        0.0           0.0
   515.000      431.336      17.575        0.0           0.0
   530.000      470.296      17.575        0.0           0.0
   545.000      505.502      17.575        0.0           0.0
   555.000      527.438       8.787        0.0           0.0
   575.000      568.553       8.787        0.0           0.0
   595.000      606.878       8.787        0.0           0.0
   600.000      616.113       2.068        0.0           0.0
   635.000      677.771       2.068        0.0           0.0
   660.000      719.318       2.068        0.0           0.0
   670.000      735.487       0.517        0.0           0.0
   705.000      790.452       0.517       54.319         0.719
   730.000      828.450       0.517      147.833         0.719
   740.000      843.406       0.517      177.412         0.719
   775.000      894.835       0.517      267.406         0.719
   800.000      930.829       0.517      324.165         0.719
   810.000      945.079       0.0        345.729         0.0
```

```
      HEIGHT OFF THE BOTTOM =        125.000 METERS


    XLO            RADIUS          PHI            RLT3           PHI3
  400.000          0.0             0.0           544.643         0.0
  465.000          0.0             0.0           633.148         0.596
  475.000          0.0             0.0           646.764         0.597
  485.000          0.0             0.0           660.380         0.598
  490.000          0.0             0.0           667.188         4.191
  500.000          0.0             0.0           680.804         4.197
  510.000          0.0             0.0           694.420         4.203
  515.000          0.0             0.0           701.228        20.428
  530.000          0.0             0.0           721.652        20.467
  545.000          0.0             0.0           742.076        20.504
  555.000          0.0             0.0           755.693        10.264
  575.000          0.0             0.0           782.925        10.286
  595.000          0.0             0.0           810.157        10.306
  600.000          0.0             0.0           816.965         2.426
  635.000          0.0             0.0           864.621         2.433
  660.000          0.0             0.0           898.661         2.438
  670.000          0.0             0.0           912.278         0.610
  705.000          0.0             0.0           959.934         0.611
  730.000          0.0             0.0           993.974         0.612
  740.000          0.0             0.0          1007.590         0.613
  775.000          0.0             0.0          1055.246         0.614
  800.000          0.0             0.0          1089.287         0.614
  810.000          0.0             0.0          1102.903         0.0



    XLO             RLT1           PHI1           RLT2           PHI2
  400.000          0.0             0.0            0.0            0.0
  465.000          0.0             0.0            0.0            0.0
  475.000          0.0             0.0            0.0            0.0
  485.000        237.512           0.396          0.0            0.0
  490.000        281.882           2.773          0.0            0.0
  500.000        344.694           2.773          0.0            0.0
  510.000        392.070           2.773          0.0            0.0
  515.000        412.549          13.469          0.0            0.0
  530.000        466.053          13.469          0.0            0.0
  545.000        511.787          13.469          0.0            0.0
  555.000        539.423           6.734          0.0            0.0
  575.000        589.940           6.734          0.0            0.0
  595.000        635.914           6.734          0.0            0.0
  600.000        646.868           1.585          0.0            0.0
  635.000        719.049           1.585          0.0            0.0
  660.000        766.966           1.585          0.0            0.0
  670.000        785.493           0.396          0.0            0.0
  705.000        848.073           0.396          0.0            0.0
  730.000        891.037           0.396          0.0            0.0
  740.000        907.893           0.396          0.0            0.0
  775.000        965.657           0.396          0.0            0.0
  800.000       1005.929           0.396        114.088          0.629
  810.000       1021.843           0.0          153.122          0.0
```

```
     HEIGHT OFF THE BOTTOM =        150.000 METERS


   XLO          RADIUS         PHI          RLT3          PHI3
400.000          0.0           0.0         614.835        0.0
465.000          0.0           0.0         714.746        0.490
475.000          0.0           0.0         730.116        0.491
485.000          0.0           0.0         745.487        0.492
490.000          0.0           0.0         753.173        3.445
500.000          0.0           0.0         768.544        3.451
510.000          0.0           0.0         783.914        3.458
515.000          0.0           0.0         791.600       16.809
530.000          0.0           0.0         814.656       16.851
545.000          0.0           0.0         837.713       16.890
555.000          0.0           0.0         853.083        8.457
575.000          0.0           0.0         883.825        8.481
595.000          0.0           0.0         914.567        8.502
600.000          0.0           0.0         922.252        2.002
635.000          0.0           0.0         976.050        2.009
660.000          0.0           0.0        1014.478        2.014
670.000          0.0           0.0        1029.848        0.504
705.000          0.0           0.0        1083.646        0.506
730.000          0.0           0.0        1122.074        0.506
740.000          0.0           0.0        1137.445        0.507
775.000          0.0           0.0        1191.243        0.508
800.000          0.0           0.0        1229.670        0.509
810.000          0.0           0.0        1245.041        0.0



   XLO           RLT1          PHI1          RLT2          PHI2
400.000          0.0           0.0           0.0           0.0
465.000          0.0           0.0           0.0           0.0
475.000          0.0           0.0           0.0           0.0
485.000          0.0           0.0           0.0           0.0
490.000          0.0           0.0           0.0           0.0
500.000        288.314         1.928         0.0           0.0
510.000        372.655         1.928         0.0           0.0
515.000        404.245         9.362         0.0           0.0
530.000        480.123         9.362         0.0           0.0
545.000        540.491         9.362         0.0           0.0
555.000        575.747         4.681         0.0           0.0
575.000        638.591         4.681         0.0           0.0
595.000        694.510         4.681         0.0           0.0
600.000        707.697         1.101         0.0           0.0
635.000        793.635         1.101         0.0           0.0
660.000        849.981         1.101         0.0           0.0
670.000        871.653         0.275         0.0           0.0
705.000        944.486         0.275         0.0           0.0
730.000        994.222         0.275         0.0           0.0
740.000       1013.686         0.275         0.0           0.0
775.000       1080.214         0.275         0.0           0.0
800.000       1126.462         0.275         0.0           0.0
810.000       1144.711         0.0           0.0           0.0
```

```
     HEIGHT OFF THE BOTTOM =        175.000 METERS


  XLO          RADIUS        PHI         RLT3        PHI3
400.000         0.0          0.0        745.304      0.0
465.000         0.0          0.0        866.416      0.360
475.000         0.0          0.0        885.048      0.361
485.000         0.0          0.0        903.681      0.362
490.000         0.0          0.0        912.997      2.536
500.000         0.0          0.0        931.630      2.542
510.000         0.0          0.0        950.263      2.548
515.000         0.0          0.0        959.579     12.388
530.000         0.0          0.0        987.528     12.427
545.000         0.0          0.0       1015.477     12.464
555.000         0.0          0.0       1034.109      6.244
575.000         0.0          0.0       1071.374      6.266
595.000         0.0          0.0       1108.640      6.286
600.000         0.0          0.0       1117.956      1.480
635.000         0.0          0.0       1183.170      1.487
660.000         0.0          0.0       1229.752      1.492
670.000         0.0          0.0       1248.384      0.373
705.000         0.0          0.0       1313.598      0.375
730.000         0.0          0.0       1360.180      0.376
740.000         0.0          0.0       1378.812      0.376
775.000         0.0          0.0       1444.026      0.377
800.000         0.0          0.0       1490.608      0.378
810.000         0.0          0.0       1509.241      0.0



  XLO           RLT1        PHI1         RLT2        PHI2
400.000         0.0         0.0          0.0         0.0
465.000         0.0         0.0          0.0         0.0
475.000         0.0         0.0          0.0         0.0
485.000         0.0         0.0          0.0         0.0
490.000         0.0         0.0          0.0         0.0
500.000       187.500       1.082        0.0         0.0
510.000       382.778       1.082        0.0         0.0
515.000       433.688       5.256        0.0         0.0
530.000       544.064       5.256        0.0         0.0
545.000       625.860       5.256        0.0         0.0
555.000       672.289       2.628        0.0         0.0
575.000       753.476       2.628        0.0         0.0
595.000       824.555       2.628        0.0         0.0
600.000       841.200       0.618        0.0         0.0
635.000       948.866       0.618        0.0         0.0
660.000      1018.887       0.618        0.0         0.0
670.000      1045.730       0.155        0.0         0.0
705.000      1135.654       0.155        0.0         0.0
730.000      1196.856       0.155        0.0         0.0
740.000      1220.771       0.155        0.0         0.0
775.000      1302.382       0.155        0.0         0.0
800.000      1359.016       0.155        0.0         0.0
810.000      1381.344       0.0          0.0         0.0
```

```
HEIGHT OFF THE BOTTOM =        197.000 METERS

   XLO          RADIUS          PHI          RLT3          PHI3
400.000          0.0            0.0        1098.289        0.0
465.000          0.0            0.0        1276.761        0.196
475.000          0.0            0.0        1304.219        0.197
485.000          0.0            0.0        1331.676        0.197
490.000          0.0            0.0        1345.404        1.384
500.000          0.0            0.0        1372.862        1.389
510.000          0.0            0.0        1400.319        1.393
515.000          0.0            0.0        1414.048        6.775
530.000          0.0            0.0        1455.233        6.803
545.000          0.0            0.0        1496.419        6.829
555.000          0.0            0.0        1523.876        3.423
575.000          0.0            0.0        1578.791        3.438
595.000          0.0            0.0        1633.705        3.453
600.000          0.0            0.0        1647.434        0.813
635.000          0.0            0.0        1743.534        0.818
660.000          0.0            0.0        1812.177        0.822
670.000          0.0            0.0        1839.635        0.206
705.000          0.0            0.0        1935.735        0.207
730.000          0.0            0.0        2004.378        0.207
740.000          0.0            0.0        2031.835        0.208
775.000          0.0            0.0        2127.936        0.208
800.000          0.0            0.0        2196.579        0.209
810.000          0.0            0.0        2224.036        0.0


   XLO           RLT1           PHI1          RLT2          PHI2
400.000          0.0            0.0           0.0           0.0
465.000          0.0            0.0           0.0           0.0
475.000          0.0            0.0           0.0           0.0
485.000          0.0            0.0           0.0           0.0
490.000          0.0            0.0           0.0           0.0
500.000          0.0            0.0           0.0           0.0
510.000        527.833          0.338         0.0           0.0
515.000        611.460          1.643         0.0           0.0
530.000        784.567          1.643         0.0           0.0
545.000        909.396          1.643         0.0           0.0
555.000        979.561          0.821         0.0           0.0
575.000       1101.499          0.821         0.0           0.0
595.000       1207.712          0.821         0.0           0.0
600.000       1232.532          0.193         0.0           0.0
635.000       1392.713          0.193         0.0           0.0
660.000       1496.637          0.193         0.0           0.0
670.000       1536.438          0.048         0.0           0.0
705.000       1669.648          0.048         0.0           0.0
730.000       1760.222          0.048         0.0           0.0
740.000       1795.598          0.048         0.0           0.0
775.000       1916.269          0.048         0.0           0.0
800.000       1999.965          0.048         0.0           0.0
810.000       2032.955          0.0           0.0           0.0
```

COMPUTER PROGRAM

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y(2,5).THTA(4),A(4,4),XLO(25),ZPRT(20),
     1SZ(25,10),SXLO(25),    SY2(25,10),SPHI(25,10),
     2SRLT1(25,10),SPHI1(25,10),SRLT2(25,10),SPHI2(25,10),
     4SRLT3(25,10),SPHI3(25,10),FY(2,5),AZDEL(1)
      F1(ZZ,Y1,Y2,DVT)=(((2*GAM/(Y2*Y2))*(Y2-(3*RTDEL/Y1)
     1*(GAM/Y2-ZZ))/(3*(D-ZZ)+2*GAM/Y2))+(1+(3*RTDEL/Y1)
     2*(GAM/(Y2*Y2)+(GAM/Y2-ZZ)*DVT/Y1)))/(3*(D-ZZ)+
     32*GAM/Y2)
      F2(ZZ,Y1,Y2)=(Y2*Y2-(3*RTDEL*(GAM-(ZZ*Y2))/Y1))/(3*Y2
     1*(D-ZZ)+2*GAM)
C     THIS PROGRAM USES THE OPTIMAL RUNGE KUTTA INTEGRATION
C     METHOD.
      READ(5,101) GRAV,BETA,ALPH,D,RTDEL,GAM,XAC
  101 FORMAT(8E10.3)
      READ(5,101) ZMAX,ZO
C     ZO IS THE DEPTH AT WHICH THE SOURCE IS LOCATED.
C     ZMAX IS THE SURFACE DEPTH--HERE IT IS ZERO.
      READ (5,101) (THTA(J),J=1,4)
      READ(5,101) ((A(I,J),J=1,4),I=1,4)
C     THE 'A' AND 'THTA' ARRAYS ARE USED FOR THE RUNGE
C     KUTTA INTEGRATION.
      WRITE(6,666)
  666 FORMAT('1')
      WRITE(6,630) GRAV,BETA,ALPH,D,RTDEL,GAM,XAC
  630 FORMAT('0',7(5X,E10.3))
      WRITE(6,633) ZMAX,ZO
  633 FORMAT(15X,' ZMAX = ',E10.3,' ZO = ',E10.3)
      WRITE(6,634) (THTA(J),J=1,4)
  634 FORMAT(4(5X,E10.3))
      WRITE(6,635) ((A(I,J),J=1,4),I=1,4)
  635 FORMAT(4E15.3)
      READ(5,130) NZDEL,NSKP,NXLO,NZPRT
C     NZDEL IS THE NUMBER OF INCREMENTS FOR INGEGRATION.
C     NSKP IS A DEGUGGING VARIABLE USED TO LIMIT THE
C     LINES OF OUTPUT.
C     NXLO IS THE NUMBER OF BUBBLES OF DIFFERENT INITIAL
C     RADII TO BE ANALYZED.
C     NZPRT IS THE NUMBER OF DEPTHS FOR WHICH COMPARISON
C     DATA IS TO BE OUTPUTTED.
  130 FORMAT(8I10)
      READ(5,101) (AZDEL(J),J=1,NZDEL)
C     AZDEL IS THE INCREMENT FOR INTEGRATION.
      WRITE(6,800) NZDEL,NSKP,NXLO,NZPRT
  800 FORMAT(15X,' NZDEL = ',I8,' NSKP = ',I8,' NXLO = ',
     1I8,' NZPRT = ',I8)
      WRITE(6,802) (AZDEL(J),J=1.NZDEL)
  802 FORMAT(' THE AZDEL ARE ',/,8E15.3)
      READ(5,101) (XLO(J),J=1,NXLO)
C     XLO IS THE INITIAL BUBBLE RADIUS.
      WRITE(6,631) (XLO(J),J=1,NXLO)
  631 FORMAT ('0','THE INITIAL RADIUS ARE',/,5X,10E12.3)
      READ(5,101) (ZPRT(J),J=1,NZPRT)
C     ZPRT IS THE DEPTH FOR WHICH COMPARISON DATA IS TO BE
C     OUTPUTTED.
      WRITE(6,632) (ZPRT(J),J=1,NZPRT)
  632 FORMAT ('0','THE INPUT HEIGHTS ARE',/,5X,10E12.3)
      GB=GRAV/BETA
C     NOW COMPUTE THE INITIAL BUBBLE VELOCITY.
      DO 80 K=1,NXLO
      DD2=DSQRT(XLO(K))
      DD3=RTDEL/DSQRT(GB/XAC)
      CALL PHIS(XLO(K),PHIO)
      VT10=GB*XLO(K)*XLO(K)/ALPH
      VT20=DSQRT(GB*XLO(K)/XAC)
      VT0=     ((-1.0+(1.0+(4.0*VT10*VT10)/(VT20*VT20))
     1**0.500000)/
     2((2.0*VT10*VT10)/(VT20*VT20)))*VT10
      VO=VTO
C     GB IS GRAV DIVIDED BY BETA.
      SXLO(K)=XLO(K)*.1D7
C     ALL VARIABLES BEGINNING WITH S ARE STORAGE ARRAYS
```

```
C        FOR THE FINAL OUTPUT.
C        NOTE THAT RADII ARE READ IN AS METERS BUT OUTPUTTED
C        IN MICRONS.
         Y(1,1)=0.
         Y(2,1)=XLO(K)
         IFLG=0
         DO 45 I=1,2
         DO 46 J=1,4
         FY(I,J)=0.
   46    CONTINUE
   45    CONTINUE
         WRITE(6,636) GB,VT0,V0,VT10,VT20
  636    FORMAT('1',2X,'GB = ',E12.5,' VT0  = ',E12.5,' V0 = ',
        1E12.5,
        2'    VT10 = ',E12.5,'    VT20 = ',E12.5)
         WRITE(6,637) (Y(J,1),J=1,2)
  637    FORMAT(10X,6(3X,E12.5))
         Z=Z0
         KK=1
         KKK=0
         HHT=0.0
C        HHT IS THE HEIGHT OFF THE BOTTOM--IT IS EQUAL TO Z-Z0.
         IS0=0
         VT=VT0
         I2=0
         ZTLDY=Z0
         WRITE(6,600) V0,SXL0(K),Z0,ZTLDY,VT0
  600    FORMAT('0',/////,20X,' V0 = ',E10.3,'  METERS/SEC  ',
        120X,
        2' L0 = ',E10.3,'  MICRONS ',/,20X,' Z0 = ',F10.3,
        3' METERS ',/,
        420X,' ZTLDY = ',F10.3,' METERS ',//,20X,' V0 TERMINAL =
        5 ',
        6E10.3,' METERS/SEC')
         WRITE(6,666)
         II=0
         IS=0
         XXL0=XL0(K)*.1D7
  701    II=II+1
         ZDEL=AZDEL(1)
         IS=IS+1
         IF(IFLG.EQ.1)  GO TO 661
   99    DO 23 J=1,4
         VT1=GB*Y(2,J)*Y(2,J)/ALPH
         VT2=DSQRT(GB*Y(2,J)/XAC)
         VTR2=VT1*VT1/(VT2*VT2)
         VT=((-1.+DSQRT(1.+4.*VTR2))/(2.*VTR2))*VT1
C        DVT IS DVT/DL (A DERIVATIVE).
         DVT=(3*VT1/Y(2,J))/(1+4*VTR2)**0.500000-VT/Y(2,J)
C        THE INTEGRATION IS DONE HERE.
         FY(1,J)=F1(Z+THTA(J)*ZDEL,VT,Y(2,J),DVT)
         FY(2,J)=F2(Z+THTA(J)*ZDEL,VT,Y(2,J))
         DO 25 N=1,2
         Y(N,J+1)=Y(N,1)+(A(J,1)*FY(N,1)+A(J,2)*FY(N,2)+A(J,3)
        1*FY(N,3)+
        2A(J,4)*FY(N,4))*ZDEL
C        WHEN THE RADIUS Y(2,J) IS LESS THAN ZERO, IFLG IS
C        SET TO ONE AND THE INTEGRATING STEPS ARE SKIPPED
C        FOR THE REST OF THE PROGRAM.
         IF(Y(N,J+1).LT.0.)  GO TO 661
   25    CONTINUE
   23    CONTINUE
         Y(1,1)=Y(1,5)
         Y(2,1)=Y(2,5)
         GO TO 662
  661    Y(1,1)=1.
         Y(2,1)=0.
         IFLG=1
         PHI=0.
  662    Z=Z+ZDEL
         IF(Z.GT..100D-01)  GO TO 999
         HHT=HHT+ZDEL
```

```
      IF(IS.EQ.NSKP)  GO TO 28
      GO TO 701
   28 IS=0
      IF(IFLG.EQ.1)  GO TO 663
      VT1=GB*Y(2,1)*Y(2,1)/ALPH
      VT2=DSQRT(GB*Y(2,1)/XAC)
      VTR2=VT1*VT1/(VT2*VT2)
      VT=((-1.+DSQRT(1.+4.*VTR2))/(2.*VTR2))*VT1
      GAM77=DEXP(Y(1,1))
      PHI=VTO*PHIO/(VT*GAM77)
C     PHI IS THE GENERAL CASE DISTRIBUTION.
  663 RZ=(D-Z)/(D-Z0)
      DD11=(D-Z0)/(D-Z)
      DD1=DSQRT((D-Z0)/(D-Z))
      RLT1=(DD11*XLO(K)*XLO(K)*XLO(K)+1.5*ALPH*RTDEL
     1*(Z*Z-Z0*Z0)/(GB*(D-Z)))
      IF(RLT1.LT..1D-18)  RLT1=0.
      RLT1=RLT1**0.333333333333
      PHI1=PHIO*RZ
C     PHI1 IS FOR THE VISCOUS DRAG MODEL.
      IF(RLT1.EQ.0.0)  PHI1=0.0
      RLT2=(DD1*DD2*DD2*DD2+DD3*((D-Z)*(1.-DD1*DD1*DD1)-3.
     1*D*(1.-DD1)))
      IF(RLT2.LT..1D-09)  RLT2=0.
      RLT2=RLT2**0.666666666666
      PHI2=PHIO*DSQRT(RZ)
C     PHI2 IS FOR THE TURBULENT DRAG, SURFACE ACTIVE
C     MATERIALS MODEL.
      IF(RLT2.EQ.0.0)  PHI2=0.0
      RZ3=RZ**0.333333333333
      RLT3=XLO(K)/RZ3
      VT13=GB*RLT3*RLT3/ALPH
      VT23=DSQRT(GB*RLT3/XAC)
      VTR3=VT13*VT13/(VT23*VT23)
      VT3=((-1.+DSQRT(1.+4.*VTR3))/(2.*VTR3))*VT13
      PHI3=PHIO*VTO*RZ3/VT3
C     PHI3 IS FOR THE NO GAS DIFFUSION CASE.
      Y2=Y(2,1)*1.D6
      XXXX=DABS(HHT-ZPRT(KK))
      RRLT1=RLT1*.1D7
      RRLT2=RLT2*.1D7
      RRLT3=RLT3*.1D7
      IF(XXXX.GT.(0.1*ZDEL))  GO TO 223
      KKK=KKK+1
      SZ(K,KKK)=Z
      SY2(K,KKK)=Y(2,1)*.1D7
      SPHI(K,KKK)=PHI
      SRLT1(K,KKK)=RLT1*.1D7
      SPHI1(K,KKK)=PHI1
      SRLT2(K,KKK)=RLT2*.1D7
      SPHI2(K,KKK)=PHI2
      SRLT3(K,KKK)=RLT3*.1D7
      SPHI3(K,KKK)=PHI3
      KK=KK+1
  223 GO TO 701
  999 CONTINUE
   80 CONTINUE
      DO 81 KKK=1,NZPRT
      WRITE(6,82)
   82 FORMAT('1')
      WRITE(6,83) ZPRT(KKK)
   83 FORMAT('0',/////,30X,' HEIGHT OFF THE BOTTOM = ',
     1F13.3,' METERS ',///)
      WRITE(6,90)
   90 FORMAT(23X,'      XLO  ','     RADIUS','      PHI   ',
     1'       RLT3 ','      PHI3 ')
      DO 85 K=1,NXLO
      WRITE(6,86)  SXLO(K),SY2(K,KKK),SPHI(K,KKK),
     1SRLT3(K,KKK),SPHI3(K,KKK)
   85 CONTINUE
      WRITE(6,92)
   92 FORMAT('0',////)
```

```
      WRITE(6,91)
   91 FORMAT(23X,'      XLO  ','       RLT1 ','      PHI1 ',
     1'      RLT2','       PHI2')
      DO 87 K=1,NXLO
      WRITE(6,86)  SXLO(K),SRLT1(K,KKK),SPHI1(K,KKK),
     1SRLT2(K,KKK),SPHI2(K,KKK)
   86 FORMAT(23X,5F11.3)
   87 CONTINUE
   81 CONTINUE
      STOP
      END


      SUBROUTINE PHIS(R,PHIO)
      IMPLICIT REAL*8 (A-H,O-Z)
C     R IS MEASURED IN METERS
      IF(R.LT..4650D-03)  GO TO 1
      IF((R.GE..4650D-03).AND.(R.LT..4900D-03))  GO TO 2
      IF((R.GE..4900D-03).AND.(R.LT..5150D-03))  GO TO 3
      IF((R.GE..5150D-03).AND.(R.LT..55500D-03))  GO TO 4
      IF((R.GE..5550D-03).AND.(R.LT..6000D-03))  GO TO 5
      IF((R.GE..6000D-03).AND.(R.LT..6700D-03))  GO TO 6
      IF((R.GE..6700D-03).AND.(R.LT..7400D-03))  GO TO 7
      IF((R.GE..7400D-03).AND.(R.LT..8100D-03))  GO TO 8
      IF(R.GE..8100D-03)  GO TO 9
    1 PHIO=0.
      GO TO 10
    2 PHIO=1.
      GO TO 10
    3 PHIO=7.
      GO TO 10
    4 PHIO=34.
      GO TO 10
    5 PHIO=17.
      GO TO 10
    6 PHIO=4.
      GO TO 10
    7 PHIO=1.
      GO TO 10
    8 PHIO=1.
      GO TO 10
    9 PHIO=0.
   10 CONTINUE
      RETURN
      END
```

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13. ABSTRACT

A bubble transport equation was developed and a formal solution obtained for the bubble distribution function. The distribution function obtained depends upon the model used for bubble drag, gas diffusion, and ocean circulation. As a specific application, solutions were obtained for a one-dimensional, quiescent, steady state ocean with a bottom source at 197 meters. The characteristic equations were integrated numerically to test the accuracy of approximate analytical solutions. The calculated bubble densities were compared with experimental data available for this problem. When gas diffusion is neglected, the theoretical and experimental distributions are in general agreement.

14 KEY WORDS	LINK A		LINK B		LINK C	
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